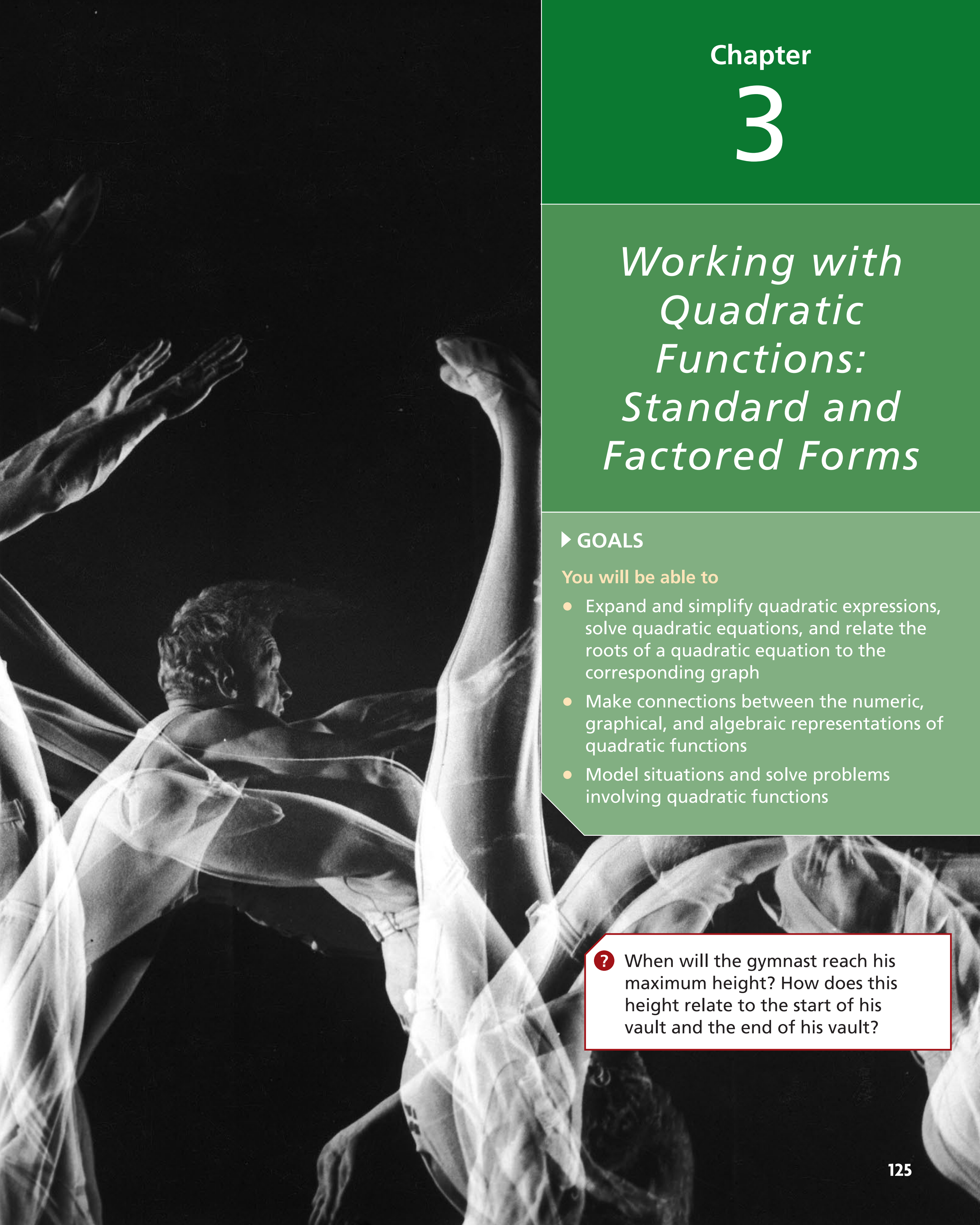


Chapter

3

Working with Quadratic Functions: Standard and Factored Forms

GOALS

You will be able to

- Expand and simplify quadratic expressions, solve quadratic equations, and relate the roots of a quadratic equation to the corresponding graph
- Make connections between the numeric, graphical, and algebraic representations of quadratic functions
- Model situations and solve problems involving quadratic functions

? When will the gymnast reach his maximum height? How does this height relate to the start of his vault and the end of his vault?

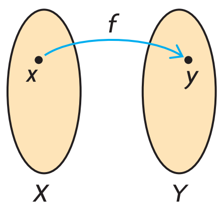
3 Getting Started

WORDS You Need to Know

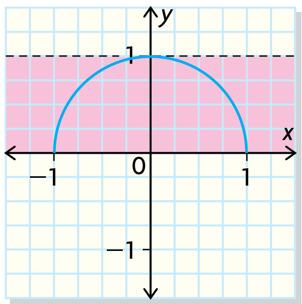
1. Match each term with its picture or example.

a) linear equation
b) zeros
c) function
d) domain
e) range
f) function notation

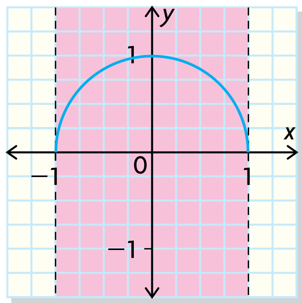
i) $f(x) = 2x^2 - 5x + 3$

iii)


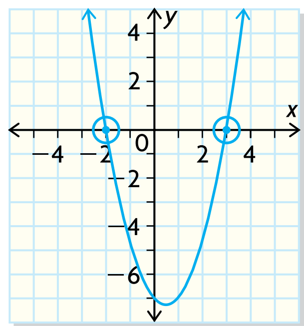
v) $y = 3x + 4$

ii)


$\{y \in \mathbf{R} \mid 0 \le y \le 1\}$

iv)


$\{x \in \mathbf{R} \mid -1 \le x \le 1\}$

vi)


$x = -2$ and $x = 3$

Study Aid

- For help, see Essential Skills Appendix, A-11.

SKILLS AND CONCEPTS You Need

Solving Linear Equations

- To solve a linear equation, isolate the variable.
- Whatever opearation you do to one side, you must do to the other.
- A linear equation has only one solution.

EXAMPLE

Solve $7y - 12 = 30$.

Solution

$$7y - 12 + 12 = 30 + 12$$

To isolate the variable y, add +12 to both sides. Simplify.

$$\frac{7y}{7} = \frac{42}{7}$$

Divide both sides by the coefficient of the variable.

$$y = 6$$

2. Solve the following linear equations.

a) $2y + 3 = 15$

b) $3x - 5 = 11$

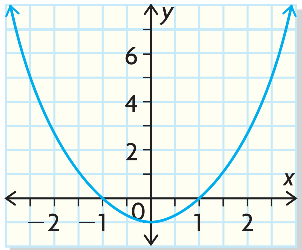
c) $-2a + 3 = -6$

d) $4c + 6 = 13$

Determining the Characteristics of a Quadratic Function from Its Graph

EXAMPLE

State the vertex, equation of the axis of symmetry, domain, and range of this function.



Study Aid

- For help, see Essential Skills Appendix, A-8.

Solution

vertex: $(0, -1)$

equation of the axis of symmetry: $x = 0$

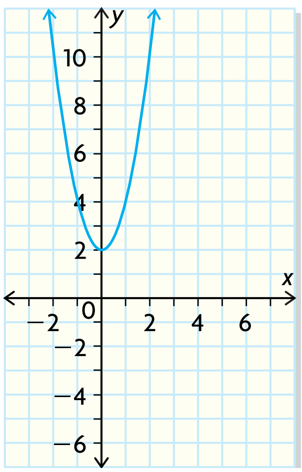
domain: $\{x \in \mathbf{R}\}$

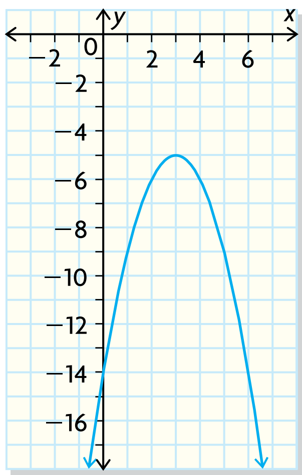
range: $\{y \in \mathbf{R} \mid y \geq -1\}$

3. For each of the following, state the vertex, equation of the axis of symmetry, domain, and range.

a) $y = 2x^2 + 2$

b) $y = -(x - 3)^2 - 5$





PRACTICE

Study ***Aid***

- For help, see Essential Skills Appendix.

Question	Appendix
4	A-9
6	A-10
7	A-7 and 8
9	A-12

4. Simplify the following.

a) $2x + 3y - 7x + 2y$
b) $3(m + 2) + 5(m - 2)$
c) $2x^2(3x^2 + 5x)$
d) $2x(x + 2) - 3(x^2 + 2x - 4)$

5. Determine the x- and y-intercepts.

a) $y = 3x - 7$
b) $2x - 6y = 12$
c) $5x = 10 - 2y$
d) $3x + 4y - 12 = 0$

6. Factor the following.

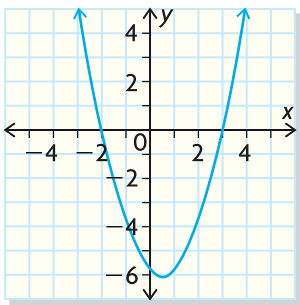
a) $5x^2 + 15 - 5x$
b) $x^2 - 11x + 10$
c) $2x^2 + 7x - 15$
d) $6x^2 + 7x - 3$
e) $2x^2 + 4x - 6$
f) $x^2 - 121$

7. Graph the following.

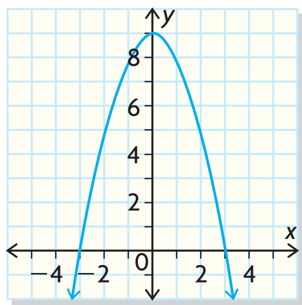
a) $y = 3x - 2$
b) $2x - 4y - 8 = 0$
c) $y = x^2 + x - 3$
d) $y = x^2 - 4$

8. For each graph, identify the x- and y-intercepts and the maximum or minimum value. State why that value is a maximum or minimum and what it tells you about a, the **coefficient** of the x^2-term of the quadratic function.

a)



b)



9. Decide whether the functions in tables (a) and (b) are linear or nonlinear. Justify your answer.

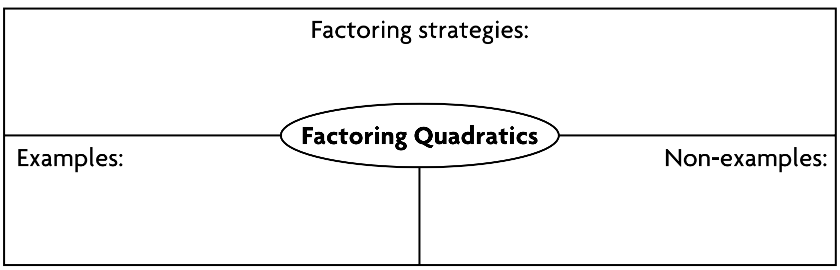
a)

x	y
0	2
1	1
2	0
3	−1
4	−2
5	−3
6	−4
7	−5
8	−6

b)

x	y
−3	15
−2	5
−1	−1
0	−3
1	−1
2	5
3	15
4	29
5	47

10. Complete the chart by writing down what you know about how and when to **factor** quadratic expressions.



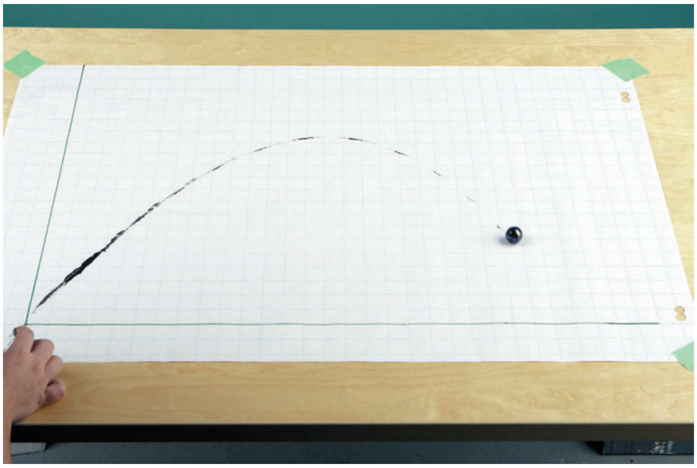
APPLYING *What You Know*

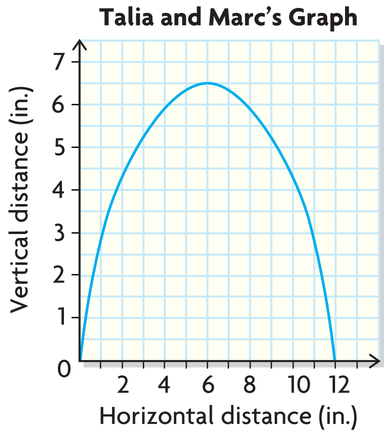
Properties of the Quadratic Function

Talia is babysitting Marc.

Talia notices that when Marc rolls a large marble on a flat surface, it travels in a straight line. Talia and Marc think that the path will look different if they roll the marble on a slanted table.

To see the path the marble travels, they dip the marble in paint and roll it on graph paper taped to a slanted table. The resulting graph is shown.





? How could you describe the graph?

A. What shape resulted? Was Talia and Marc's hypothesis correct?

B. What are the x-intercepts, or zeros, of the graph?

C. What was the maximum distance travelled by the marble from the edge of the table?

D. When was this maximum distance reached?

E. What is the equation of the axis of symmetry of the curve? Explain how this relates to the x-intercepts (zeros) of the function.

F. What is the vertex of the graph? Explain how it is related to the location of the maximum.

G. What factors do you think could affect the size and shape of the parabola made by the rolling marble?

3.1 Exploring a Situation Using a Quadratic Function

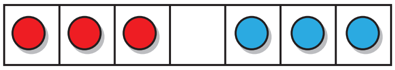
- chart graph paper
- two different-coloured counters

GOAL

Use quadratic functions to model real-world applications.

EXPLORE the Math

In the game of leapfrog, you move red and blue counters along a line of spaces in a board. Equal numbers of red and blue counters are placed in the spaces opposite each other and separated by an empty space in the middle.

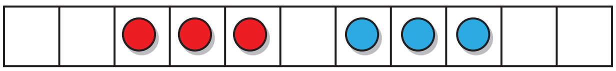


The object of the game is to move the red counters into the spaces occupied by the blue counters, and vice versa.

The rules allow two types of moves:
- a move to an adjacent space
- a jump over a counter of the other colour

? What is the minimum number of moves needed to switch seven blue and seven red counters?

A. Play the game by drawing a grid on paper with at least 11 boxes. The grid must contain an odd number of boxes, and you must use equal numbers of red and blue counters.



B. In a table, relate the number of counters of each colour, N, to the minimum number of moves, M, required to switch the counters.

Number of Counters of each Colour, *N*	1	2	3	4	5
Minimum Number of Moves, *M*					

C. Consult other groups to see if your minimum number of moves is the same as theirs.

D. Create a scatter plot of the data you collected. Identify the independent and dependent variables.

E. Determine the type of graph represented by your data.

F. What type of equation could be used to model your data?

Reflecting

G. Predict the minimum number of moves needed to switch seven red and seven blue counters.

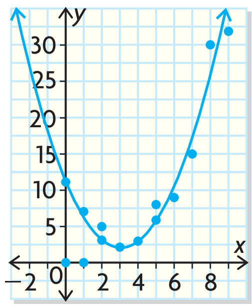
H. Holly thinks that the function $f(x) = x^2 + 2x$ models the minimum number of moves required to switch equal numbers of counters. What do x and $f(x)$ represent in this situation? Is Holly correct? Explain.

I. What are the domain and range of $f(x)$? Is it necessary to restrict the domain? Explain.

In Summary

Key Ideas

- A scatter plot that appears to have a parabolic shape or part of a parabolic shape might be modelled by a quadratic function.
- The domain and range of the quadratic model may need to be restricted for the situation you are dealing with.



Need to Know

- The domain and range may need to be limited to whole numbers if the data are **discrete**; that is, if fractions, decimals, and negative numbers are not valid for the situation.

FURTHER Your Understanding

1. Play leapfrog with six counters of each colour, and determine the minimum number of moves. Does the algebraic model predict the minimum number of moves? Explain.

2. Play the game again, increasing N, the number of counters used each time you play. This time, for each move, keep track of the colour of the counter and the type of move made.

For example, with one red and one blue counter, the moves could be red slide (RS), blue jump (BJ), and then red slide (RS). Set up a table as shown to record your results. Identify any patterns you see in your table.

N			
1	RS	BJ	RS
2			
3			

3. Play the game with one side having one more counter than the other side. Can a quadratic model be used to relate the minimum number of moves to the lower number of counters? Explain.

3.2 Relating the Standard and Factored Forms

YOU WILL NEED

- graph paper
- graphing calculator (optional)

GOAL

Compare the factored form of a quadratic function with its standard form.

Communication *Tip*

A company's revenue is its money earned from sales. A revenue function, $R(x)$, is the product of the number of items sold and price.

Revenue = (number of items sold)(price)

LEARN ABOUT the Math

To raise money, some students sell T-shirts. Based on last year's sales, they know that

- they can sell 40 T-shirts a week at $10 each
- if they raise the price by $1, they will sell one less T-shirt each week

They picked some prices, estimated the number of shirts they might sell, and calculated the revenue.

Price ($)	T-shirts Sold	Revenue ($)
10	40	$10 \times 40 = 400$
$10 + 15 = 25$	$40 - 15 = 25$	$25 \times 25 = 625$
$10 + 30 = 40$	$40 - 30 = 10$	$40 \times 10 = 400$

Rachel and Andrew noticed that as they increased the price, the revenue increased and then decreased. Based on this pattern, they suggested quadratic functions to model revenue, where x is the number of $1 increases.

- Rachel's function is $R(x) = (40 - x)(10 + x)$.
- Andrew's function is $R(x) = -x^2 + 30x + 400$.

? What is the maximum revenue they can earn on T-shirt sales?

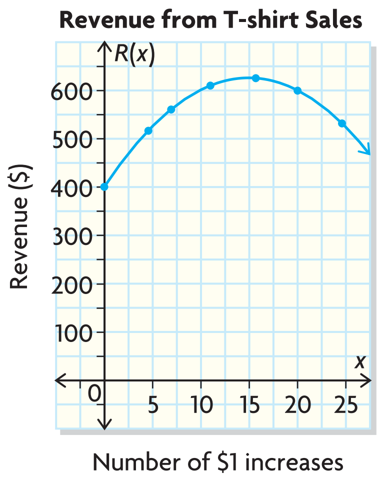
EXAMPLE 1 Connecting the standard and factored forms of quadratic functions

Compare the two functions:
$R(x) = (40 - x)(10 + x)$ and $R(x) = -x^2 + 30x + 400$

Enrique's Solution: Comparing and Analyzing the Graphs

Rachel's Function	
x	$R(x) = (40 - x)(10 + x)$
0	$= (40 - 0)(10 + 0)$ $= (40)(10)$ $= 400$
4	504
8	576
12	616
16	624
20	600
24	544

Andrew's Function	
x	$R(x) = -x^2 + 30x + 400$
0	$= -0^2 + 30(0) + 400$ $= 0 + 0 + 400$ $= 400$
4	504
8	576
12	616
16	624
20	600
24	544



The tables of values are the same for both functions, so the graphs will be the same. I need only one graph.

I used Rachel's table of values to make a graph.

The two functions are equivalent.

The maximum revenue is about $625.

To get the maximum revenue, I looked at my graph and found the y-coordinate of the vertex. Since the vertex occurs at $(15, 625)$, $R(15) = 625$.

factored form

a quadratic function in the form $f(x) = a(x - r)(x - s)$

standard form

a quadratic function in the form $f(x) = ax^2 + bx + c$

Rachel's revenue function is in **factored form**. If you expand and simplify it to express it in **standard form**, then you can compare both functions to see if they are the same.

Beth's Solution: Comparing and Analyzing the Functions

$$R(x) = (40 - x)(10 + x)$$
$$R(x) = 400 + 40x - 10x - x^2$$
$$R(x) = 400 + 30x - x^2$$

I multiplied the binomials and collected like terms. Rachel's revenue function in standard form is the same as Andrew's revenue function, but the terms are in a different order.

$$R(x) = -x^2 + 30x + 400$$

I rearranged my function to match Andrew's.

zeros of a relation

the values of x for which a relation has the value zero. The zeros of a relation correspond to the x-intercepts of its graph

$$0 = -x^2 + 30x + 400 \text{ or}$$
$$0 = (40 - x)(10 + x)$$
$$40 - x = 0 \quad \text{and} \quad 10 + x = 0$$
$$40 = 0 + x \quad \text{and} \quad x = 0 - 10$$
$$40 = x \quad \text{and} \quad x = -10$$

The x-intercepts, or **zeros**, occur when the revenue is 0, so $R(x) = 0$. I used Rachel's function to find the x-intercepts by setting each factor equal to zero and solving for x.

$$x = \frac{(40 + (-10))}{2}$$
$$x = \frac{30}{2}$$
$$x = 15$$

The maximum value is the y-coordinate of the vertex, and the vertex lies on the axis of symmetry.

To find the axis of symmetry, I added the zeros and divided by 2.

$$R(15) = (40 - 15)(10 + 15)$$
$$R(15) = (25)(25)$$
$$R(15) = 625$$

To find the maximum revenue I substituted the x-value into Rachel's equation.

The maximum revenue is $625.

Reflecting

A. If there is no increase in the cost of the T-shirts, what will the revenue be? Whose revenue model can be used to see this value most directly?

B. Which revenue function would you use to determine the maximum value? Explain.

APPLY the Math

If you are given a quadratic function in standard form and it is possible to write it in factored form, you can use the factored form to determine the coordinates of the vertex.

EXAMPLE 2 Selecting a factoring strategy to determine the vertex of a quadratic function

Determine the coordinates of the vertex of $f(x) = 2x^2 - 5x - 12$, and sketch the graph of $f(x)$.

Talia's Solution

$$f(x) = 2x^2 - 5x - 12$$
$$f(x) = 2x^2 - 8x + 3x - 12$$

First, I factored the trinomial by decomposition. I found two numbers that multiply to give -24 (from 2×-12) and add to give -5. The numbers are -8 and 3. I replaced the middle term with $-8x$ and $3x$.

$$f(x) = (2x^2 - 8x) + (3x - 12)$$
$$f(x) = 2x(x - 4) + 3(x - 4)$$
$$f(x) = (x - 4)(2x + 3)$$
$$0 = (x - 4)(2x + 3)$$
$$x - 4 = 0 \quad \text{and} \quad 2x + 3 = 0$$
$$x = 0 + 4 \quad \text{and} \quad 2x = 0 + (-3)$$
$$x = 4 \quad \text{and} \quad x = -\frac{3}{2} \text{ or } -1.5$$

I grouped the first two terms and the second two terms. I divided out the common factor $(x - 4)$ from both groups. I used the factors to find the zeros.

$$x = \frac{4 + (-1.5)}{2}$$

$$x = \frac{2.5}{2}$$

$$x = 1.25$$

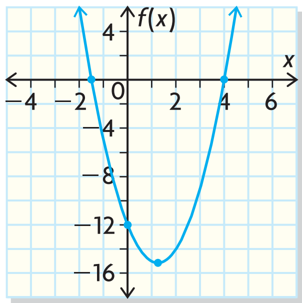
I used the zeros to find the axis of symmetry. I know that the axis of symmetry is halfway between the zeros. I added the two zeros and divided by 2.

The equation of the axis of symmetry is $x = 1.25$.

$$\begin{aligned} f(1.25) &= 2(1.25)^2 - 5(1.25) - 12 \\ &= 3.125 - 6.25 - 12 \\ &= -15.125 \end{aligned}$$

I put $x = 1.25$ into the original function and solved to get the y-coordinate of the vertex.

The vertex is $(1.25, -15.125)$.



I used the values I calculated to draw the graph of a parabola. It opens up, since $a > 0$.

The vertex is $(1.25, -15.125)$.

The zeros are $(4, 0)$ and $(-1.5, 0)$.

The y-intercept is $(0, -12)$.

EXAMPLE 3 Solving problems using a quadratic function model

The height of a football kicked from the ground is given by the function $h(t) = -5t^2 + 20t$, where $h(t)$ is the height in metres and t is the time in seconds from its release.

a) Write the function in factored form.
b) When will the football hit the ground?
c) When will the football reach its maximum height?
d) What is the maximum height the football reaches?
e) Graph the height of the football in terms of time without using a table of values.

Matt's Solution

a) $h(t) = -5t^2 + 20t$

$h(t) = -5t(t - 4)$

I wrote the function in factored form by dividing out the common factor $-5t$.

b) $0 = -5t(t - 4)$

$-5t = 0 \quad \text{and} \quad t - 4 = 0$

$t = 0 \quad \text{and} \quad t = 4$

When the football hits the ground, the height is zero. $h(t) = 0$ when each factor is 0. The zeros are 0 and 4. The football is on the ground to start and then returns 4 s later.

c) $t = 0$ and $t = 4$ are the zeros.

$$t = \frac{0 + 4}{2}$$

$t = 2$

The ball reaches its maximum height when $t = 2$ s.

I know that the maximum value is the y-coordinate of the vertex, and it lies on the axis of symmetry, which is halfway between the zeros. The coefficient of x^2 is -5, which means that the parabola opens down. Therefore, there is a maximum value. The equation $t = 2$ is the axis of symmetry.

d) $h(t) = -5t^2 + 20t$

$h(2) = -5(2)^2 + 20(2)$

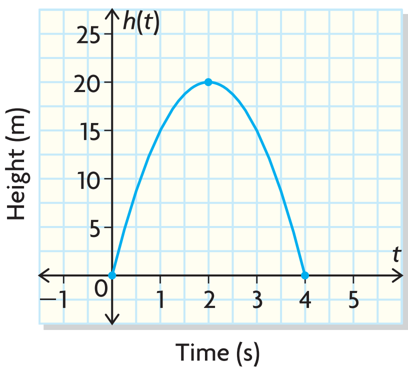
$= -20 + 40$

$= 20$

The maximum height is 20 m.

I know the ball reaches the maximum height at 2 s. So I calculated $h(2)$, the maximum height.

e)

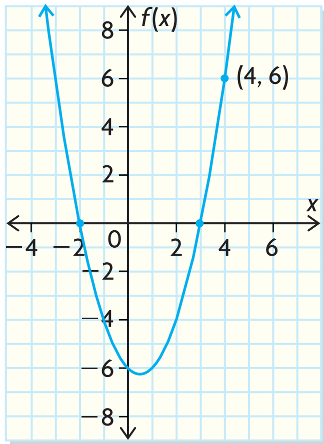


I marked the two zeros and the vertex that I found on graph paper. Since $a < 0$, those three points are part of a parabola that opens down, so I joined them by sketching a parabola. I can see from the graph that the y-intercept is 0. This is also seen in the function when it's written in standard form: $h(t) = -5t^2 + 20t + 0$.

If you are given the graph of a quadratic function and you can determine the zeros and the coordinates of another point on the parabola, you can determine the equation of the function in factored and standard forms.

EXAMPLE 4 Representing a quadratic function in factored and standard forms from a graph

From the graph of this quadratic function, determine the function's factored and standard forms.



Randy's Solution

$$f(x) = a(x - r)(x - s)$$
$$f(x) = a(x - (-2))(x - (+3))$$
$$f(x) = a(x + 2)(x - 3)$$

A quadratic function can be written in the form $f(x) = a(x - r)(x - s)$, where r and s are the zeros of the function and a indicates the shape of the parabola. From the graph, the zeros are $x = -2$ and $x = 3$. I substituted -2 for r and 3 for s, then simplified.

Let $(x, y) = (4, 6)$
$$6 = a(4 + 2)(4 - 3)$$
$$6 = a(6)(1)$$
$$a = 1$$

To get a, I picked any point on the curve, substituted its coordinates into the function, and then solved for a.

$$f(x) = 1(x + 2)(x - 3)$$
$$f(x) = 1(x^2 - 3x + 2x - 6)$$
$$f(x) = x^2 - x - 6$$

I used my value of a to rewrite the function in factored form.
I expanded to get the standard form.

From the graph the y-intercept of the parabola is -6. This is also seen in the function when it's written in standard form.

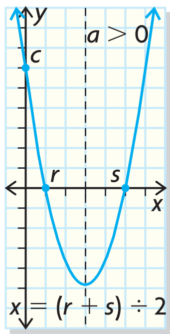
In Summary

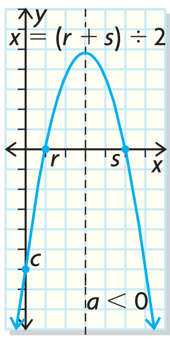
Key Ideas

- Some quadratic functions in standard form, $f(x) = ax^2 + bx + c$, can be expressed in factored form as $f(x) = a(x - r)(x - s)$ by factoring. The two forms of the quadratic function are equivalent.
- All quadratic functions in factored form can be expressed in standard form by expanding. The two forms of the quadratic function are equivalent.

Need to Know

- Both the standard and factored forms provide useful information for graphing the parabola.
  - When a quadratic function is expressed in factored form, $f(x) = a(x - r)(x - s)$, r and s are the x-intercepts, or zeros, of the function. The axis of symmetry is the vertical line that runs through the midpoint of the zeros and is defined by $x = (r + s) \div 2$. This value is also the x-coordinate of the vertex.





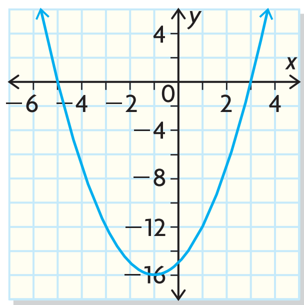
  - When a quadratic function is expressed in standard form, $f(x) = ax^2 + bx + c$, c is the y-intercept of the function.
  - The value of a in both factored and standard forms determines the direction that the parabola opens. If $a > 0$, the parabola open up; if $a < 0$, the parabola opens down.

CHECK Your Understanding

1. The parabola shown is congruent to $y = x^2$.

a) What are the zeros of the function?

b) Write the equation in factored form.



2. Express each quadratic function in factored form. Then determine the zeros, the equation of the axis of symmetry, and the coordinates of the vertex.

a) $f(x) = 2x^2 + 12x$

b) $f(x) = x^2 - 7x + 12$

c) $f(x) = -x^2 + 100$

d) $f(x) = 2x^2 + 5x - 3$

3. Express each quadratic function in standard form. Determine the y-intercept.

a) $f(x) = 3x(x - 4)$

b) $f(x) = (x - 5)(x + 7)$

c) $f(x) = 2(x - 4)(3x + 2)$

d) $f(x) = (3x - 4)(2x + 5)$

Communication | *Tip*

Two parabolas that are congruent have exactly the same size and shape.

PRACTISING

4. For each quadratic function, determine the zeros, the equation of the axis of symmetry, and the coordinates of the vertex without graphing.

a) $g(x) = 2x(x + 6)$
b) $g(x) = (x - 8)(x + 4)$
c) $g(x) = (x - 10)(2 - x)$
d) $g(x) = (2x + 5)(9 - 2x)$
e) $g(x) = (2x + 3)(x - 2)$
f) $g(x) = (5 - x)(5 + x)$

5. Express each function in factored form. Then determine the zeros, the equation of the axis of symmetry, and the coordinates of the vertex without graphing.

a) $g(x) = 3x^2 - 6x$
b) $g(x) = x^2 + 10x + 21$
c) $g(x) = x^2 - x - 6$
d) $g(x) = 3x^2 + 12x - 15$
e) $g(x) = 2x^2 - 13x - 7$
f) $g(x) = -6x^2 + 24$

6. Match the factored form on the left with the correct standard form on the right. How did you decide on your answer?

a) $y = (2x + 3)(x - 4)$
b) $y = (4 - 3x)(x + 3)$
c) $y = (3x - 4)(x - 3)$
d) $y = (3 - 4x)(4 - x)$
e) $y = (x + 3)(3x - 4)$

i) $y = 4x^2 - 19x + 12$
ii) $y = -3x^2 - 5x + 12$
iii) $y = 2x^2 - 5x - 12$
iv) $y = 3x^2 - 13x + 12$
v) $y = 3x^2 + 5x - 12$

7. Determine the maximum or minimum value for each quadratic function.

a) $f(x) = (7 - x)(x + 2)$
b) $f(x) = (x + 5)(x - 9)$
c) $f(x) = (2x + 3)(8 - x)$
d) $g(x) = x^2 + 7x + 10$
e) $g(x) = -x^2 + 25$
f) $g(x) = 4x^2 + 4x - 3$

8. K Graph each quadratic function by hand by determining the zeros, vertex, axis of symmetry, and y-intercept.

a) $g(x) = (2x - 1)(x - 4)$
b) $f(x) = (3x - 1)(2x - 5)$
c) $f(x) = x^2 - x - 20$
d) $g(x) = 2x^2 + 2x - 12$
e) $f(x) = -x^2 - 2x + 24$
f) $f(x) = -4x^2 - 16x + 33$

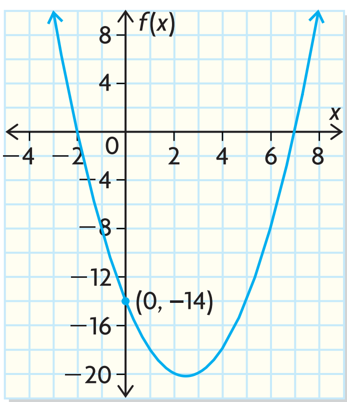
9. **a)** When a quadratic function is in standard form, what information about the graph can be easily determined? Provide an example.
b) When a quadratic function is in factored form, what information about the graph can be easily determined? Provide an example.

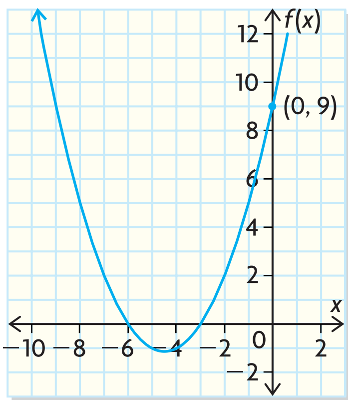
10. Graph each function, and complete the table.

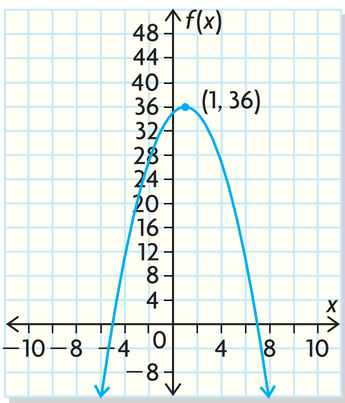
	Factored Form	Standard Form	Axis of Symmetry	Zeros	y-intercept	Vertex	Maximum or Minimum Value
a)	$R(x) = (40 - x)(10 + x)$	$R(x) = -x^2 + 30x + 400$					
b)	$f(x) = (x - 4)(x + 2)$						
c)		$g(x) = -x^2 + 2x + 8$					
d)	$p(x) = (3 - x)(x + 1)$						
e)		$j(x) = 4x^2 - 121$					

11. **A** The height of water, $h(t)$, in metres, from a garden hose is given by the function $h(t) = -5t^2 + 15t$, where t is time in seconds. Express the function in factored form, and then use the zeros to determine the maximum height reached by the water.

12. For each graph, write the equation in both factored and standard forms.

a)


c)


b)


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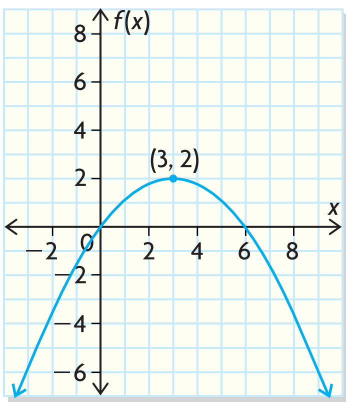
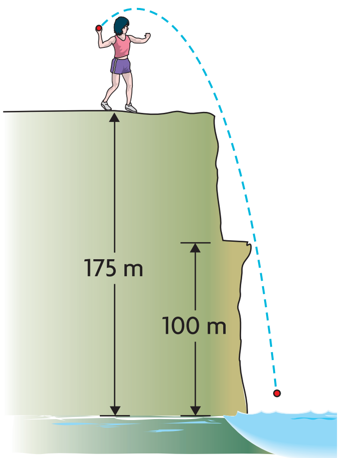


13. Complete the table.

	Zeros	Axis of Symmetry	Maximum or Minimum Value	Vertex	Function in Factored Form	Function in Standard Form
a)	2 and 8		6			
b)	−7 and −2		−2			
c)	−1 and 9		5			
d)	−8 and 0		−5			



14. A ball is thrown into water from a cliff that is 175 m high. The height of the ball above the water after it is thrown is modelled by the function $h(t) = -5t^2 + 10t + 175$, where $h(t)$ is the height in metres and t is time in seconds.

a) When will the ball reach the water below the cliff?

b) When will the ball reach a ledge that is 100 m above the water?

15. T The safe stopping distance for a boat travelling at a constant speed in calm water is given by $d(v) = 0.002(2v^2 + 10v + 3000)$, where $d(v)$ is the distance in metres and v is the speed in kilometres per hour. What is the initial speed of the boat if it takes 30 m to stop?

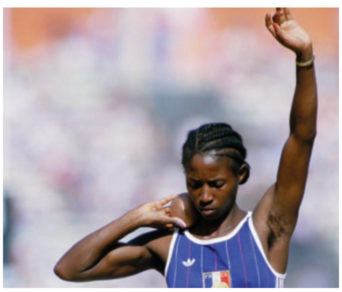
16. C Describe how you would sketch the graph of $f(x) = 2x^2 - 4x - 30$ without using a table of values.

Extending

17. The stainless-steel Gateway Arch in St. Louis, Missouri, is almost parabolic in shape. It is 192 m from the base of the left leg to the base of the right leg. The arch is 192 m high. Determine a function, in standard form, that models the shape of the arch.

18. A model rocket is shot straight up into the air. The table shows its height, $h(t)$, at time t. Determine a function, in factored form, that estimates the height of the rocket at any given time.

Time (s)	0	1	2	3	4	5	6
Height (m)	0.0	25.1	40.4	45.9	41.6	27.5	3.6



19. The path of a shot put is given by $h(d) = 0.0502(d^2 - 20.7d - 26.28)$, where $h(d)$ is the height and d is the horizontal distance, both in metres.

a) Rewrite the relation in the form $h(d) = a(d - r)(d - s)$, where r and s are the zeros of the relation.

b) What is the significance of r and s in this question?

3.3 Solving Quadratic Equations by Graphing

GOAL

Use graphs to solve quadratic equations.

YOU WILL NEED

- graph paper
- graphing calculator

INVESTIGATE the Math

A model rocket is launched from the roof of a building.

The height, $h(t)$, in metres, at any time, t, in seconds, is modelled by the function $h(t) = -5t^2 + 15t + 20$.

? When will the rocket hit the ground?

A. Use a graphing calculator to graph the height function. Determine the zeros, the axis of symmetry, and the vertex of this function.

B. What is the rocket's height when it hits the ground? Use this value to write a quadratic equation you could solve to determine when the rocket hits the ground.

C. Substitute one of the zeros you determined in part A into the quadratic equation you wrote in part B. Repeat this for the other zero. What do you notice?

D. What are the roots of the quadratic equation you wrote in part B?

E. State the domain and range of the function in the context of this problem.

F. What is the starting height of the rocket? Where is it on the graph? Where is it in the function?

G. When will the rocket hit the ground?

quadratic equation

an equation that contains a polynomial whose highest degree is 2; for example, $x^2 + 7x + 10 = 0$

root of an equation

a number that when substituted for the unknown, makes the equation a true statement

for example, $x = 2$ is a root of the equation $x^2 - x - 2 = 0$ because $2^2 - 2 - 2 = 0$

the root of an equation is also known as a *solution* to that equation

Reflecting

H. How are the zeros of the function $h(t)$ related to the roots of the quadratic equation $0 = -5t^2 + 15t + 20$?

I. The quadratic equation $0 = -5t^2 + 15t + 20$ has two roots. Explain why only one root is an acceptable solution in this situation. How does your explanation relate to the domain of the function $h(t)$?

J. How can the graph of the function $f(x) = ax^2 + bx + c$ help you solve the quadratic equation $ax^2 + bx + c = 0$?

APPLY the Math

EXAMPLE 1 Connecting graphs to the solutions of a quadratic equation

Determine the solutions of the quadratic equation $x^2 - 8x + 12 = -3$ by graphing. Check your solutions.

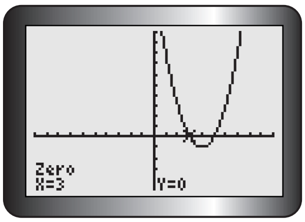
Nash's Solution: Rearranging the Equation to Determine the Zeros on the Corresponding Graph

$x^2 - 8x + 12 = -3$

$x^2 - 8x + 15 = 0$

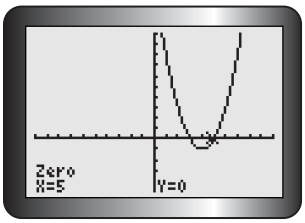
$g(x) = x^2 - 8x + 15$

I rearranged the equation so that it was equal to zero. I used the corresponding function, $g(x) = x^2 - 8x + 15$, to solve the equation by finding its zeros.



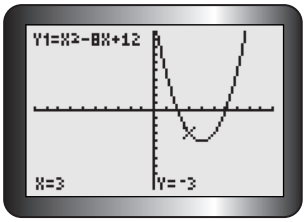
I graphed $g(x) = x^2 - 8x + 15$. I used the zero operation on my calculator to see where the graph crosses the x-axis.

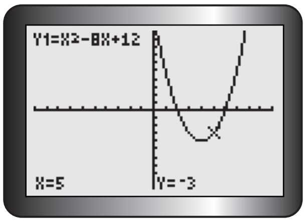
One solution is $x = 3$.



There's another point where the graph crosses the x-axis, so I used the zero operation again.

The other solution is $x = 5$.





To check, I entered the left side of the original equation $(x^2 - 8x + 12)$ into Y1, and then used the value operation to see if the y-value is equal to -3.

Tech Support

For help determining values and zeros using a graphing calculator, see Technical Appendix, B-3 and B-8.

Both solutions give a value of -3, so I know they're correct.

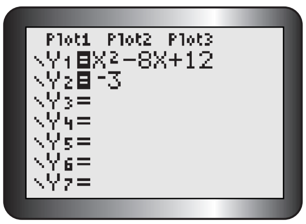
Rearranging a quadratic equation into the form $ax^2 + bx + c$ and then graphing $f(x) = ax^2 + bx + c$ is one method you can use to solve a quadratic equation. Another method is to treat each side of the equation as a function and then graph both to determine the point(s) of intersection.

Lisa's Solution: Determining the Points of Intersection of Two Corresponding Functions

$$x^2 - 8x + 12 = -3$$

$$f(x) = x^2 - 8x + 12 \quad \text{and} \quad g(x) = -3$$

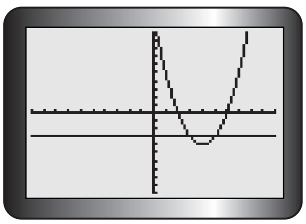
I separated the equation into two functions, using each side of the equation. I called the left side $f(x)$ and the right side $g(x)$.

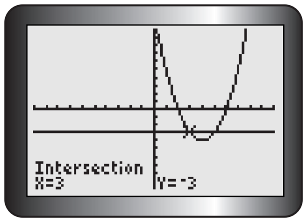


The solutions of the original equation correspond to the points that lie on both graphs at the same time. I can determine these solutions by graphing both functions and locating the points of intersection.

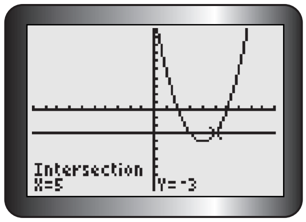
Tech Support

For help determining points of intersection using a graphing calculator, see Technical Appendix, B-11.





I found the points of intersection by using the intersect operation on the calculator. The x-coordinates of these points are the solutions of the original equation.



The two solutions are $x = 3$ and $x = 5$.

Check:

L.S. $= x^2 - 8x + 12$ R.S. $= -3$

When $x = 3$,

$$\begin{aligned} \text{L.S.} &= (3)^2 - 8(3) + 12 \\ &= 9 - 24 + 12 \\ &= -3 \\ &= \text{R.S.} \end{aligned}$$

When $x = 5$,

$$\begin{aligned} \text{L.S.} &= (5)^2 - 8(5) + 12 \\ &= 25 - 40 + 12 \\ &= -3 \\ &= \text{R.S.} \end{aligned}$$

I substituted each of my answers into the original equation to see if I had the correct solutions. They both work, so I know my solutions are correct.

EXAMPLE 2 Selecting strategies to make predictions from a quadratic equation

The population of an Ontario city is modelled by the function $P(t) = 0.5t^2 + 10t + 300$, where $P(t)$ is the population in thousands and t is the time in years. *Note*: $t = 0$ corresponds to the year 2000.

a) What was the population in 2000?
b) What will the population be in 2010?
c) When is the population expected to be 1 050 000?

Guillaume's Solution

a)
$$\begin{aligned} P(0) &= 0.5(0)^2 + 10(0) + 300 \\ &= 0 + 0 + 300 \\ &= 300 \end{aligned}$$

The year 2000 corresponds to $t = 0$, so I set $t = 0$ in the equation and evaluated. I remembered to multiply the solution by 1000 because P is in thousands.

The population was 300 000 in the year 2000.

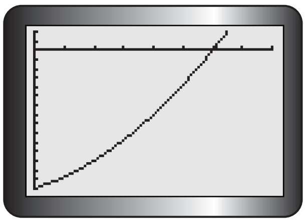
b)
$$\begin{aligned} P(10) &= 0.5(10)^2 + 10(10) + 300 \\ &= 0.5(100) + 100 + 300 \\ &= 50 + 100 + 300 \\ &= 450 \end{aligned}$$

The year 2010 corresponds to $t = 10$, so I set $t = 10$ in the equation. Again, I had to remember to multiply the solution by 1000 because P is in thousands.

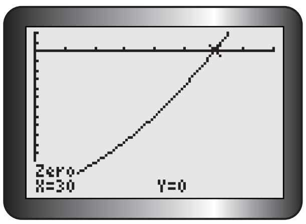
The population will be 450 000 in the year 2010.

c)
$$\begin{aligned} 1050 &= 0.5t^2 + 10t + 300 \\ 0 &= 0.5t^2 + 10t - 750 \end{aligned}$$

To see when the population will be 1 050 000, I let P be 1050 because P is in thousands, and $\frac{1\,050\,000}{1000} = 1050$.



I rearranged the equation to get zero on the left side. I used my graphing calculator to graph $P(t) = 0.5t^2 + 10t - 750$, and then I looked for the zeros. I decided to use $t \geq 0$ as the domain, so I'll have only one zero.



Using the zero operation on my graphing calculator, I got 30 for the zero.

The population should reach 1 050 000 in the year 2030.

Since $t = 0$ is used to represent the year 2000, I added 30 to get 2030.

EXAMPLE 3 Solving problems involving a quadratic equation

The function $h(t) = 2 + 50t - 1.862t^2$, where $h(t)$ is the height in metres and t is time in seconds, models the height of a golf ball above the planet Mercury's surface during its flight.

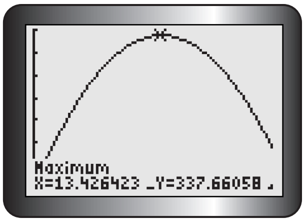
a) What is the maximum height reached by the ball?
b) How long will the ball be above the surface of Mercury?
c) When will it reach a height of 200 m on the way down?

Candace's Solution

Tech *Support*

For help determining maximum and minimum values using a graphing calculator, see Technical Appendix, B-9.

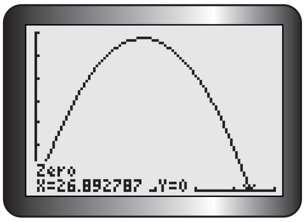
a)



I graphed the equation $y = 2 + 50t - 1.862t^2$ on my graphing calculator. I used the maximum operation on my calculator to get the maximum value.

The maximum height of the golf ball is about 337.7 m.

b)



When the ball reaches a height of 0 m, it's no longer above the planet. So I need to identify the zeros.

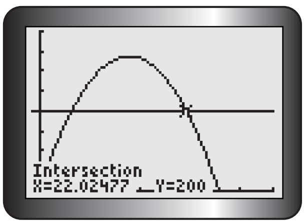
The first zero is before the golf ball has been hit. I want the second zero.

The ball will be above the planet's surface for about 26.9 s.

Tech *Support*

For help determining points of intersection using a graphing calculator, see Technical Appendix, B-11.

c) $2 + 50t - 1.862t^2 = 200$



To find the time when the ball is at a height of 200 m, I set $h(t)$ equal to 200. To solve this equation, I graphed $y = 2 + 50t - 1.862t^2$ and $y = 200$ on the same axes. From the graph, I can see that there are two points of intersection. The second point is the height on the way down, so I found it using the intersection operation on the calculator.

The ball will reach a height of 200 m on its way down at about 22.0 s.

In Summary

Key Idea

- All quadratic equations can be expressed in the form $ax^2 + bx + c = 0$ using algebraic techniques. They can be solved by graphing the corresponding function $f(x) = ax^2 + bx + c$. The zeros, or x-intercepts, of the function are the solutions, or roots, of the equation.

Need to Know

- A quadratic equation is any equation that contains a polynomial whose highest degree is 2. For example, $x^2 + 8x + 15 = 0$.
- An alternative to solving $ax^2 + bx + c = d$ is to graph both $y = ax^2 + bx + c$ and $y = d$. The solutions will be those points where the two functions intersect.
- You should substitute the solutions into the original equation to verify the result.

CHECK Your Understanding

1. Graph each function by hand. Then use the graph to solve the corresponding quadratic equation.

a) $g(x) = -x^2 + 5x + 14$ and $-x^2 + 5x + 14 = 0$

b) $f(x) = x^2 + 8x + 15$ and $x^2 + 8x + 15 = 0$

2. Graph each function using a graphing calculator. Then use the graph to solve the corresponding quadratic equation.

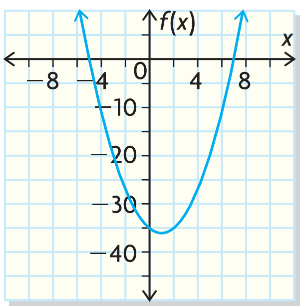
a) $h(x) = x^2 - x - 20$ and $x^2 - x - 20 = 0$

b) $f(x) = x^2 - 5x - 9$ and $x^2 - 5x - 9 = 0$

PRACTISING

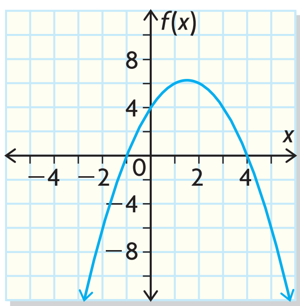
3. For each function, write the corresponding quadratic equation whose solutions are also the zeros of the function.

a)



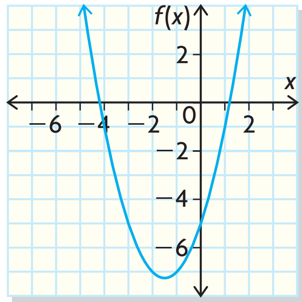
$y = x^2 - 2x - 35$

b)



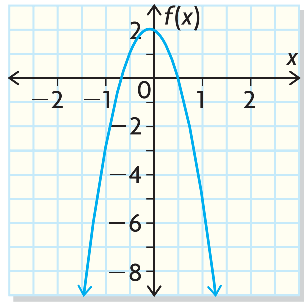
$y = -x^2 + 3x + 4$

c)



$y = x^2 + 3x - 5$

d)



$y = -6x^2 - x + 2$

4. Graph the corresponding function to determine the roots of each equation. Verify your solutions.

a) $x^2 - 8x = -16$

b) $2x^2 + 3x - 20 = 0$

c) $-5x^2 + 15x = 10$

d) $x^2 + 4x = 8$

e) $x^2 + 5 = 0$

f) $4x^2 - 64 = 0$

5. K Graph each function. Then use the graph to solve the quadratic equation.

a) $p(x) = 3x^2 + 5x - 2$ and $3x^2 + 5x - 2 = 0$

b) $f(x) = 2x^2 - 11x - 21$ and $2x^2 - 11x - 21 = 0$

c) $p(x) = 8x^2 + 2x - 3$ and $8x^2 + 2x - 3 = 0$

d) $f(x) = 3x^2 + x + 1$ and $3x^2 + x + 1 = 0$

6. The population, $P(t)$, of an Ontario city is modelled by the function $P(t) = 14t^2 + 650t + 32\,000$. *Note:* $t = 0$ corresponds to the year 2000.

a) What will the population be in 2035?

b) When will the population reach 50 000?

c) When was the population 25 000?

7. The function $h(t) = 2.3 + 50t - 1.86t^2$ models the height of an arrow shot from a bow on Mars, where $h(t)$ is the height in metres and t is time in seconds. How long does the arrow stay in flight?

8. The height of an arrow shot by an archer is given by the function $h(t) = -5t^2 + 18t - 0.25$, where $h(t)$ is the height in metres and t is time in seconds. The centre of the target is in the path of the arrow and is 1 m above the ground. When will the arrow hit the centre of the target?

9. The student council is selling cases of gift cards as a fundraiser. The revenue, $R(x)$, in dollars, can be modelled by the function $R(x) = -25x^2 + 100x + 1500$, where x is the number of cases of gift cards sold. How many cases must the students sell to maximize their revenue?

10. The Wheely Fast Co. makes custom skateboards for professional riders. The company models its profit with the function $P(b) = -2b^2 + 14b - 20$, where b is the number of skateboards produced, in thousands, and $P(b)$ is the company's profit, in hundreds of thousands of dollars.
A

a) How many skateboards must be produced for the company to break even?

b) How many skateboards does Wheely Fast Co. need to produce to maximize profit?

Communication Tip

A company's break-even point is the point at which the company shows neither a profit nor a loss. This occurs when the profit is zero.

11. A ball is tossed upward from a cliff that is 40 m above water. The height of the ball above the water is modelled by $h(t) = -5t^2 + 10t + 40$, where $h(t)$ is the height in metres and t is the time in seconds. Use a graph to answer the following questions.

a) What is the maximum height reached by the ball?

b) When will the ball hit the water?

12. The cost, $C(n)$, in dollars, of operating a concrete-cutting machine is modelled by $C(n) = 2.2n^2 - 66n + 655$, where n is the number of minutes the machine is in use.

a) How long must the machine be in use for the operating cost to be a minimum?

b) What is the minimum cost?

13. a) For each condition, determine an equation in standard form of a quadratic function that
T

i) has two zeros

ii) has one zero

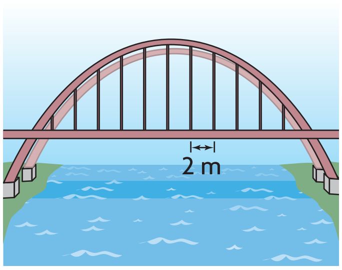
iii) has no zeros

b) What is the maximum number of zeros that a quadratic function can have? Explain.

14. a) What quadratic function could be used to determine the solution of the quadratic equation $3x^2 - 2x + 5 = 4$?
C

b) Explain how you could use the function in part (a) to determine the solutions of the equation.

Extending



15. A parabolic arch is used to support a bridge. Vertical cables are every 2 m along its length. The table of values shows the length of the cables with respect to their placement relative to the centre of the arch. Negative values are to the left of the centre point. Write an algebraic model that relates the length of each cable to its horizontal placement.

Distance from Centre of Arch (m)	Length of Cable (m)
−10	120.0
−8	130.8
−6	139.2
−4	145.2
−2	148.8
0	150.0
2	148.8
4	145.2
6	139.2
8	130.8
10	120.0

16. Determine the points of intersection of

a) $y = 4x - 1$ and $y = 2x^2 + 5x - 7$

b) $y = x^2 - 2x - 9$ and $y = -x^2 + 5x + 6$

17. Show that the function $y = 2x^2 - 3x + 4$ cannot have a y-value less than 1.5.

3 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How are the factored and standard forms of a quadratic function related?

A: The factored form and the standard form are different algebraic representations of the same function. They have the same zeros, axis of symmetry, and maximum or minimum values. As a result, they have the same graph.

Study *Aid*

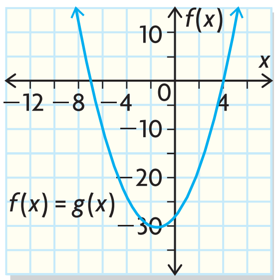
- See Lesson 3.2, Example 1.
- Try Mid-Chapter Review Question 5.

EXAMPLE

$f(x) = (x - 4)(x + 7)$ and $g(x) = x^2 + 3x - 28$

Expand the factored form to get the standard form.

$$\begin{aligned} f(x) &= (x-4)(x+7) \\ &= x^2 + 7x - 4x - 28 \\ &= x^2 + 3x - 28 \\ &= g(x) \end{aligned}$$



Factor the standard form to get the factored form.

$$\begin{aligned} g(x) &= x^2 + 3x - 28 \\ &= (x+7)(x-4) \\ &= f(x) \end{aligned}$$

Q: What information can you determine about a parabola from the factored and standard forms of a quadratic function?

A: From the factored form, $f(x) = a(x - r)(x - s)$, you can determine

- the zeros, or x-intercepts, which are r and s
- the equation of the axis of symmetry, which is $x = (r + s) \div 2$
- the coordinates of the vertex by substituting the value of the axis of symmetry for x in the function

From the standard form, $f(x) = ax^2 + bx + c$, you can determine

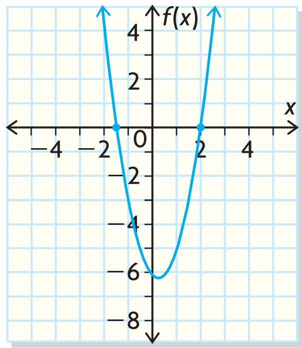
- the y-intercept, which is c

From both forms you can determine

- the direction in which the parabola opens: up when $a > 0$ and down when $a < 0$

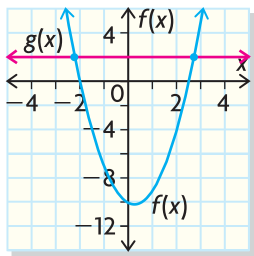
Q: How do you solve a quadratic equation by graphing?

A1: If the quadratic equation is in the form $ax^2 + bx + c = 0$, then graph the function $f(x) = ax^2 + bx + c$ to see where the graph crosses the x-axis. The roots, or solutions, of the equation are the zeros, or x-intercepts, of the function.



A2: If the quadratic equation is not in the form $ax^2 + bx + c = 0$, for example, $ax^2 + bx + c = d$, then either

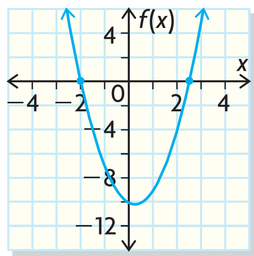
graph the functions $f(x) = ax^2 + bx + c$ and $g(x) = d$ and see where they intersect,



In this case, the solutions to the equation $ax^2 + bx + c = d$ are the points of intersection of the graphs.

or

rewrite $ax^2 + bx + c = d$ in the form $ax^2 + bx + c - d = 0$ and then graph the function $f(x) = ax^2 + bx + c - d$ to see where it crosses the x-axis.



In this case, the solutions to the equation $ax^2 + bx + c = d$ are the zeros of the graph.

PRACTICE Questions

Lesson 3.2

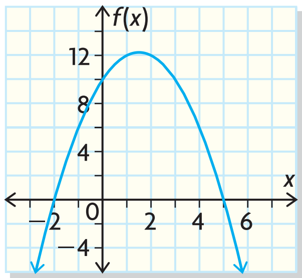
1. Write each function in standard form.

a) $f(x) = (x + 7)(2x + 3)$
b) $g(x) = (6 - x)(3x + 2)$
c) $f(x) = -(2x + 3)(4x - 5)$
d) $g(x) = -(5 - 3x)(-2x + 1)$

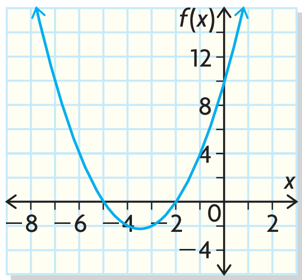
2. Match each function to its graph.

a) $f(x) = x^2 + 7x + 10$
b) $f(x) = (5 - x)(x + 2)$
c) $f(x) = -x^2 + 3x + 10$
d) $f(x) = (x + 2)(x + 5)$
e) $f(x) = x^2 - 3x - 10$

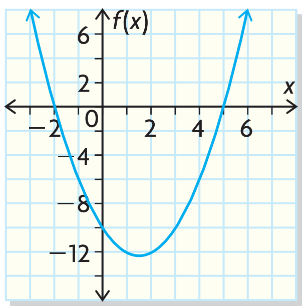
i)



ii)



iii)



3. Determine the maximum or minimum of each function.

a) $f(x) = x^2 + 2x - 35$
b) $g(x) = -2x^2 - 6x + 36$
c) $f(x) = 2x^2 - 9x - 18$
d) $g(x) = -2x^2 + x + 15$

4. Which form of a quadratic function do you find most useful? Use an example in your explanation.

Lesson 3.3

5. Graph each function, and then use the graph to locate the zeros.

a) $f(x) = x^2 + 2x - 8$
b) $g(x) = 15x^2 - 2x - 1$
c) $f(x) = 8x^2 + 6x + 1$
d) $g(x) = 2x^2 - 3x - 5$

6. Solve by graphing.

a) $x^2 + 2x - 15 = 0$
b) $(x + 3)(2x + 5) = 0$
c) $2x^2 - x - 6 = 0$
d) $x^2 + 7x = -12$

7. A field-hockey ball must stay below waist height, approximately 1 m, when shot; otherwise, it is a dangerous ball. Sally hits the ball. The function $h(t) = -5t^2 + 10t$, where $h(t)$ is in metres and t is in seconds, models the height of the ball. Has she shot a dangerous ball? Explain.

8. Are $x = 3$ and $x = -4$ the solutions to the equation $x^2 - 7x + 12 = 0$? Explain how you know.

3.4 Solving Quadratic Equations by Factoring

GOAL

Use different factoring strategies to solve quadratic equations.

LEARN ABOUT the Math

The profit of a skateboard company can be modelled by the function $P(x) = -63 + 133x - 14x^2$, where $P(x)$ is the profit in thousands of dollars and x is the number of skateboards sold, also in thousands.

? When will the company break even, and when will it be profitable?

EXAMPLE 1 Selecting a factoring strategy to determine the break-even points and profitability

Determine when the company is profitable by calculating the break-even points.

Tim's Solution

$$-63 + 133x - 14x^2 = 0$$

The company will break even when the profit is 0, so I set the profit function equal to zero. I can also write the function in factored form, then I can determine the zeros without having to graph the function.

$$-14x^2 + 133x - 63 = 0$$

I rewrote the equation in the form $ax^2 + bx + c = 0$. It's easier to see if I can factor it in standard form.

$$-7(2x^2 - 19x + 9) = 0$$

I divided out the common factor of -7 because this makes it easier to factor the trinomial.

$$-7(2x^2 - 18x - 1x + 9) = 0$$
$$-7[(2x(x - 9) - 1(x - 9)] = 0$$
$$-7(x - 9)(2x - 1) = 0$$

I used decomposition to factor the trinomial. I need two numbers that multiply to give 18 (from 2×9) and add to give -19. The two numbers are -18 and -1. I rewrote the middle term as two terms: $-18x$ and $-1x$. I grouped the terms and divided out the common factor of $(x - 9)$.

$$x - 9 = 0 \quad \text{or} \quad 2x - 1 = 0$$

Since the zeros of the function $f(x) = -14x^2 + 133x - 63$ are the roots of the equation $-14x^2 + 133x - 63 = 0$, I determined these numbers by setting each factor equal to zero and solving for x without having to make a graph first.

$$x = 9 \quad \text{or} \quad 2x = 1$$
$$x = \frac{1}{2}$$

The solutions of the equations are 9 and $\frac{1}{2}$. These numbers represent the number of skateboards sold in the thousands, so I multiplied the solutions by 1000.

The company will break even when it sells 500 or 9000 skateboards. The company will be profitable if it sells between 500 and 9000 skateboards.

Reflecting

A. If a quadratic equation is expressed in factored form, how can you determine the solutions?

B. Why does equating each of the factors of a quadratic equation to zero enable you to determine its solutions?

APPLY the Math

Since some quadratic equations can be solved by factoring, you will need to use a suitable factoring strategy based on the type of quadratic in each equation.

EXAMPLE 2 Selecting factoring strategies to solve quadratic equations

Solve by factoring.

a) $x^2 - 2x - 24 = 0$ **b)** $9x^2 - 25 = 0$ **c)** $4x^2 + 20x = -25$

Damir's Solution

a) $x^2 - 2x - 24 = 0$

$(x - 6)(x + 4) = 0$

This equation involves a trinomial where $a = 1$. First, I found two numbers that multiply to give -24 and add to give -2. The two numbers are -6 and 4.

$x - 6 = 0$ and $x + 4 = 0$

$x = 6$ and $x = -4$

I set each factor equal to zero and solved.

b) $9x^2 - 25 = 0$

$(3x - 5)(3x + 5) = 0$

This equation involves a **difference of squares** because $9x^2 = (3x)^2$, $25 = 5^2$, and the terms are separated by a minus sign. A difference of squares in the form $a^2 - b^2$ factors to $(a - b)(a + b)$.

$3x - 5 = 0$ and $3x + 5 = 0$

$3x = 5$ and $3x = -5$

$x = \frac{5}{3}$ and $x = -\frac{5}{3}$

I set each factor equal to zero and solved.

c) $4x^2 + 20x = -25$

$4x^2 + 20x + 25 = 0$

First, I wrote the equation in the form $ax^2 + bx + c = 0$.

This equation involves a trinomial where $a \neq 1$.

I factored it by decomposition, so I found two numbers that multiply to give 100 (from 4×25) and add to give 20. The numbers are 10 and 10. Then I rewrote the $20x$-term as $10x + 10x$.

$4x^2 + 10x + 10x + 25 = 0$

$2x(2x + 5) + 5(2x + 5) = 0$

$(2x + 5)(2x + 5) = 0$

Now I can factor by grouping in pairs and divide out the common factor of $(2x + 5)$.

$2x + 5 = 0$

$2x = -5$

$x = -\frac{5}{2}$

Since both factors are the same, I set one factor equal to zero and solved. This equation has two solutions that are the same, so it really has only one solution.

Some quadratic equations may require some algebraic manipulation before you can solve by factoring.

EXAMPLE 3 Reasoning to solve a more complicated quadratic equation

Solve $2(x+3)^2 = 5(x+3)$. Verify your solution.

Beth's Solution: Expanding and Rearranging the Equation

$$2(x+3)^2 = 5(x+3)$$
$$2(x+3)(x+3) = 5(x+3)$$
$$2(x^2+3x+3x+9) = 5x+15$$
$$2(x^2+6x+9) = 5x+15$$
$$2x^2+12x+18 = 5x+15$$
$$2x^2+12x+18-5x-15 = 0$$

I expanded both sides by multiplying. I wrote $(x+3)^2$ as $(x+3)(x+3)$ to make it easier to expand. Next, I multiplied each expression using the distributive property, then collected like terms and rewrote the equation so that the right side is equal to zero. It's easier to find the factors when one side of the equation is equal to zero.

$$2x^2+7x+3 = 0$$
$$2x^2+6x+1x+3 = 0$$
$$2x(x+3)+1(x+3) = 0$$
$$(x+3)(2x+1) = 0$$

I used decomposition to factor because the coefficient of x^2 is not 1. I need two numbers that multiply to give 6 (from 2×3) and add to give 7. The two numbers are 6 and 1, so I rewrote $7x$ as $6x + 1x$. I grouped the terms and divided out the common factor $(x+3)$.

$$x+3=0 \quad \text{and} \quad 2x+1=0$$
$$x=-3 \quad \text{and} \quad 2x=-1$$
$$x=-\frac{1}{2}$$

I set each factor equal to zero and solved.

Then I checked my solutions.

To see if my solutions are correct, I substituted each x-value into the original equation. I checked to see if the left side of the equation and the right side of the equation give the same answer.

Check:

When $x = -3$,

$$\begin{aligned} \text{L.S.} &= 2(x+3)^2 & \text{R.S.} &= 5(x+3) \\ &= 2(-3+3)^2 & &= 5(-3+3) \\ &= 2(0)^2 & &= 5(0) \\ &= 2(0) = 0 & &= 0 \\ &= \text{R.S.} \end{aligned}$$

Both my solutions are correct.

When $x = -\frac{1}{2}$,

$$\begin{aligned} \text{L.S.} &= 2\left(-\frac{1}{2}+3\right)^2 & \text{R.S.} &= 5\left(-\frac{1}{2}+3\right) \\ &= 2\left(-\frac{1}{2}+\frac{6}{2}\right)^2 & &= 5\left(-\frac{1}{2}+\frac{6}{2}\right) \\ &= 2\left(\frac{5}{2}\right)^2 & &= 5\left(\frac{5}{2}\right) \\ &= 2\left(\frac{25}{4}\right) & &= \frac{25}{2} \\ &= \frac{25}{2} \\ &= \text{R.S.} \end{aligned}$$

Steve's Solution: Dividing out the Common Factor

$$2(x+3)^2 = 5(x+3)$$
$$2(x+3)^2 - 5(x+3) = 0$$
$$(x+3)[2(x+3) - 5] = 0$$
$$(x+3)(2x+6-5) = 0$$
$$(x+3)(2x+1) = 0$$
$$x+3 = 0 \quad \text{and} \quad 2x+1 = 0$$
$$x = 0 - 3 \quad \text{and} \quad 2x = 0 - 1$$
$$x = -3 \quad \text{and} \quad 2x = -1$$
$$x = -\frac{1}{2}$$

Since an equation is easier to solve when one side is equal to zero, I moved $5(x+3)$ to the left side.

I noticed that $(x+3)$ is common to both terms, so I divided out this common factor.

I simplified the expression in the second brackets by expanding and collecting like terms.

To solve, I set each factor equal to zero.

EXAMPLE 4 Solving a problem involving a quadratic equation

The path a dolphin travels when it rises above the ocean's surface can be modelled by the function $h(d) = -0.2d^2 + 2d$, where $h(d)$ is the height of the dolphin above the water's surface and d is the horizontal distance from the point where the dolphin broke the water's surface, both in feet. When will the dolphin reach a height of 1.8 feet?

Tyson's Solution

$$-0.2d^2 + 2d = 1.8$$

The height the dolphin will reach is 1.8 feet, so I set $h(d) = 1.8$.

$$-0.2d^2 + 2d - 1.8 = 0$$

Then I wrote the equation in the form $ax^2 + bx + c = 0$.

$$-0.2(d^2 - 10d + 9) = 0$$
$$-0.2(d-9)(d-1) = 0$$

First I divided out the common factor 0.2. Then I found two numbers that multiply to give 9 and add to give -10. The numbers are -1 and -9.

$$d - 9 = 0 \quad \text{and} \quad d - 1 = 0$$
$$d = 9 \quad \text{and} \quad d = 1$$

I set the factors equal to zero and solved.

The dolphin will reach a height of 1.8 feet twice: at a horizontal distance of 1 foot on the way up and at a horizontal distance of 9 feet on the way down.

In Summary

Key Idea

- If a quadratic equation of the form $ax^2 + bx + c = 0$ can be written in factored form, then the solutions of the quadratic equation can be determined by setting each of the factors to zero and solving the resulting equations.

Need to Know

- All quadratic equations of the form $ax^2 + bx + c = d$ must be expressed in the form $ax^2 + bx + (c - d) = 0$ before factoring. Doing this is necessary because the zeros of the corresponding function, $f(x) = ax^2 + bx + (c - d)$, are the roots of the equation $ax^2 + bx + c = d$.
- When factoring quadratic equations in the form $ax^2 + bx + c = 0$, apply the same strategies you used to factor quadratic expressions. These strategies include looking for
  - a common factor
  - a pair of numbers r and s, where $rs = ac$ and $r + s = b$
  - familiar patterns such as a difference of squares or perfect squares
- Not all quadratic expressions are factorable. As a result, not all quadratic equations can be solved by factoring. To determine whether $ax^2 + bx + c = 0$ is factorable, multiply a and c. If two numbers can be found that multiply to give the product ac and also add to give b, then the equation can be solved by factoring. If not, then factoring cannot be used. If this is the case, then other methods must be used.

CHECK Your Understanding

1. Solve.

a) $(x + 3)(x - 5) = 0$

b) $5(x - 6)(x - 9) = 0$

c) $(2x + 1)(3x - 5) = 0$

d) $2x(x - 3) = 0$

2. Solve by factoring.

a) $x^2 + x - 20 = 0$

b) $x^2 = 36$

c) $x^2 + 12x = -36$

d) $x^2 = 10x$

3. Determine whether the given number is a root of the quadratic equation.

a) $x = 2; x^2 + 6x - 16 = 0$

b) $x = -4; 2x^2 - 5x - 35 = 0$

c) $x = 1; 6x^2 + 7x = x^2 + 12$

d) $x = -1; 5x^2 + 7x = 2x^2 - 6$

PRACTISING

4. Solve by factoring. Verify your solutions.

K

a) $x^2 - 3x - 54 = 0$
b) $x^2 - 169 = 0$
c) $x^2 + 14x + 49 = 0$
d) $x^2 - 17x + 42 = 0$
e) $2x^2 - 9x - 5 = 0$
f) $3x^2 + 11x - 4 = 0$

5. Solve by factoring. Verify your solutions.

a) $x^2 = 289$
b) $9x^2 - 30x = -25$
c) $x^2 - 5x = -3x + 15$
d) $2x^2 + 3x = 16x + 7$
e) $4x^2 - 5x = 2x^2 - x + 30$
f) $x^2 + 3x + 10 = 3x^2 - 4x - 5$

6. Solve by factoring. Verify your solutions.

a) $3x(x - 2) = 4x(x + 1)$
b) $2x^2(x + 3) = -4x^2(x - 1)$
c) $(x + 5)^2 - 6 = (x + 5)$
d) $(x + 3)(x - 1) = 2(x - 5)(x + 3)$
e) $3(x - 5)^2 = x - 5$
f) $x^3 + 4x^2 = x^3 - 2x^2 - 17x - 5$

7. Solve by factoring. Verify your solutions.

a) $x^2 + 8x = 9x + 42$
b) $2(x + 1)(x - 4) = 4(x - 2)(x + 2)$
c) $x^2 + 5x - 36 = 0$
d) $3(x - 2)^2 = 2(x - 2)$
e) $8x^2 - 3x = -17x - 3$
f) $-8x^2 + 5x = 2x^2 - 8x - 3$

8. A model airplane is shot into the air. Its path is approximated by the function $h(t) = -5t^2 + 25t$, where $h(t)$ is the height in metres and t is the time in seconds. When will the airplane hit the ground?

9. The area of a rectangular enclosure is given by the function $A(w) = -2w^2 + 48w$, where $A(w)$ is the area in square metres and w is the width of the rectangle in metres.

a) What values of w give an area of 0?
b) What is the maximum area of the enclosure?

10. Snowy's Snowboard Co. manufactures snowboards. The company uses the function $P(x) = 324x - 54x^2$ to model its profit, where $P(x)$ is the profit in thousands of dollars and x is the number of snowboards sold, in thousands.

a) How many snowboards must be sold for the company to break even?
b) How many snowboards must be sold for the company to be profitable?

11. A rock is thrown down from a cliff that is 180 m high. The function
A $h(t) = -5t^2 - 10t + 180$ gives the approximate height of the rock above the water, where $h(t)$ is the height in metres and t is the time in seconds. When will the rock reach a ledge that is 105 m above the water?

12. A helicopter drops an aid package. The height of the package above the ground at any time is modelled by the function $h(t) = -5t^2 - 30x + 675$, where $h(t)$ is the height in metres and t is the time in seconds. How long will it take the package to hit the ground?

13. The manager of a hardware store knows that the weekly revenue function for batteries sold can be modelled with $R(x) = -x^2 + 10x + 30\,000$, where both the revenue, $R(x)$, and the cost, x, of a package of batteries are in dollars. According to the model, what is the maximum revenue the store will earn?

14. Kool Klothes has determined that the revenue function for selling
T x thousand pairs of shorts is $R(x) = -5x^2 + 21x$. The cost function $C(x) = 2x + 10$ is the cost of producing the shorts.

a) Write a profit function.

b) How many pairs of shorts must the company sell in order to break even?

15. Can factoring always be used to solve quadratic equations? Explain.

16. What are the advantages and disadvantages of solving quadratic
C equations by factoring?

Extending

17. A hot-air balloon drops a sandbag. The table shows the height of the sandbag at different times. When will the sandbag reach the ground?

Time (s)	0	1	2	3	4	5	6	7	8	9	10
Height (m)	1000	995	980	955	920	875	820	755	680	595	500

18. What numbers could t be if t^2 must be less than $12t - 20$?

3.5 Solving Problems Involving Quadratic Functions

YOU WILL NEED

- graphing calculator

GOAL

Select and apply factoring and graphing strategies to solve applications involving quadratics.

LEARN ABOUT the Math

A computer software company models the profit on its latest video game using the function $P(x) = -2x^2 + 32x - 110$, where x is the number of games, in thousands, that the company produces and $P(x)$ is the profit, in millions of dollars.

? How can you determine the maximum profit the company can earn?

EXAMPLE 1 Selecting a strategy to solve the problem

Choose a strategy to determine the maximum profit possible.

Matt's Solution: Using a Table of Values

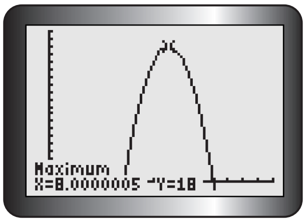
x	$P(x)$
0	−110
2	−54
4	−14
5	0
6	10
7	16
8	18
9	16

I used different values for x and then picked the highest number in the $P(x)$ column for the maximum profit.

The highest value of $P(x)$ is 18, and it occurs when $x = 8$.

The company must sell 8000 games to earn a profit of \$18 million.

Tony's Solution: Using a Graphing Calculator



I used a graphing calculator. I entered $-2x^2 + 32x - 110$ into Y1 and then used the maximum operation.

According to the graphing calculator, the maximum is 18 when $x = 8$. At first I thought that meant that the company must sell 8 games to earn a profit of \$18, and that's not much profit. Then I remembered that the function is for x thousand games and $P(x)$ is in millions of dollars.

The number of games they must produce to make a maximum profit is 8000. The profit will be \$18 million.

Donica's Solution: Factoring

$$P(x) = -2x^2 + 32x - 110$$

This is an equation of a parabola that opens downward, since the coefficient of x^2 is negative.

$$P(x) = -2(x - 5)(x - 11)$$

$$x - 5 = 0 \quad \text{and} \quad x - 11 = 0$$

$$x = 5 \quad \text{and} \quad x = 11$$

The maximum value is at the vertex, which is halfway between the function's two zeros. So I factored $P(x)$ to find the zeros. I set each factor equal to zero to solve.

The maximum occurs at

$$x = \frac{5 + 11}{2}$$

$$x = 8$$

The maximum is halfway between the zeros. I added the zeros and divided by 2.

$$P(8) = -2(8^2) + 32(8) - 110$$

$$= 18$$

I put 8 into the profit function to get $P(8) = 18$.

The company must sell 8000 games to earn a profit of \$18 million.

Tech Support

For help determining the maximum or minimum value using a graphing calculator, see Technical Appendix, B-9.

Reflecting

A. Can Matt always be certain he has determined the maximum value using his method?

B. Will Donica always be able to use her method to determine the maximum (or minimum) value of a function? Explain.

C. How do you know that each student has determined the maximum profit and that no other maximum could exist?

D. Why is finding the domain and range important for quadratic equations that model real-world situations?

E. How do you choose your strategy? What factors will affect the method you choose to solve a problem?

APPLY the Math

The strategy you use to solve a problem involving a quadratic function depends on what you are asked to determine, the quadratic function you are given, and the degree of accuracy required.

EXAMPLE 2 Selecting a tool or strategy to solve a quadratic equation

Sally is standing on the top of a river slope and throws a ball. The height of the ball at a given time is modelled by the function $h(t) = -5t^2 - 10t + 250$, where $h(t)$ is the height in metres and t is time in seconds. When will the ball be 10 m above the ground?

Rachel's Solution: Factoring

$-5t^2 - 10t + 250 = 10$

I set the original function equal to 10 because I wanted to know when the height is 10 m.

$-5t^2 - 10t + 240 = 0$

Then I rearranged the equation to get zero on the right side.

$-5(t^2 + 2t - 48) = 0$

$-5(t + 8)(t - 6) = 0$

$t + 8 = 0$ and $t - 6 = 0$

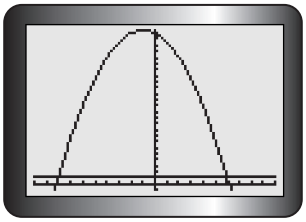
$t = -8$ and $t = 6$

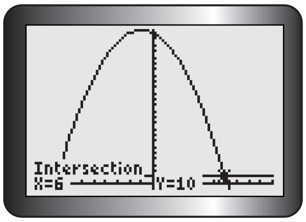
I factored and set each factor equal to zero and solved for t to find the roots.

Only one of the roots makes sense. The domain for the function is $t \geq 0$, since it doesn't make sense for time to be negative. This means that the only solution is $t = 6$.

The ball will be 10 m above the ground 6 s into its flight.

Stephanie's Solution: Using a Graphing Calculator





I entered $-5x^2 - 10x + 250$ into Y1 and 10 into Y2. I remembered to change the window setting so that I could see the whole parabola.

I noticed that the line $y = 10$ crosses the parabola twice. This means that I have to find two intersection points.

I used the intersection operation on the calculator to find the points of intersection. The intersections occur when $x = 6$ and when $x = -8$. I can use only the positive solution because the domain is $t \geq 0$.

6 s after the ball is initially thrown, it has a height of 10 m.

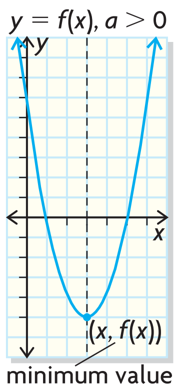
In Summary

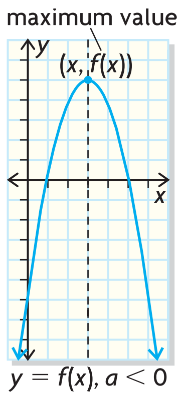
Key Ideas

- Problems involving quadratic functions can be solved with different strategies, such as
  - a table of values
  - graphing (with or without graphing technology)
  - an algebraic approach involving factoring
- The strategy used depends on what you need to determine and whether or not an estimate or a more accurate answer is required.

Need to Know

- The value of a in the quadratic equation determines the direction in which the parabola opens and whether there is a maximum or a minimum. If $a > 0$, the parabola opens up and there will be a minimum. If $a < 0$, the parabola opens down and there will be a maximum.
- The quadratic function can be used to determine the maximum or minimum value and/or the zeros of the quadratic model. These can then be used as needed to interpret or solve the situation presented.





CHECK Your Understanding

1. Solve by a table of values, a graphing calculator, and factoring: A computer software company models its profit with the function $P(x) = -x^2 + 13x - 36$, where x is the number of games, in hundreds, that the company sells and $P(x)$ is the profit, in thousands of dollars.

a) How many games must the company sell to be profitable?

b) Which method do you prefer? Why?

2. Use factoring to solve the following problem: The height of a ball above the ground is given by the function $h(t) = -5t^2 + 45t + 50$, where $h(t)$ is the height in metres and t is time in seconds. When will the ball hit the ground?

PRACTISING

3. The function $d(s) = 0.0056s^2 + 0.14s$ models the stopping distance of a car, $d(s)$, in metres, and the speed, s, in kilometres per hour. What is the speed when the stopping distance is 7 m? Use a graph to solve.

4. The population of a city is modelled by $P(t) = 14t^2 + 820t + 52\ 000$, where t is time in years. *Note:* $t = 0$ corresponds to the year 2000. According to the model, what will the population be in the year 2020? Here is Beverly's solution:

Beverly's Solution

$$\begin{aligned} P(2020) &= 14(2020)^2 + 820(2020) + 52\ 000 \\ &= 57\ 125\ 600 + 1\ 656\ 400 + 52\ 000 \\ &= 58\ 834\ 000 \end{aligned}$$

The population will be 58 834 000.

I substituted 2020 into the function for t because that's the year for which I want to know the population. Then I solved.

Are Beverly's solution and reasoning correct? Explain.

5. **K** Solve $4x^2 - 10x - 24 = 0$ using three different methods.

6. The population of a city is modelled by the function $P(t) = 0.5t^2 + 10t + 200$, where $P(t)$ is the population in thousands and t is time in years. *Note:* $t = 0$ corresponds to the year 2000. According to the model, when will the population reach 312 000?

7. **T** Which methods can be used to solve $-4.9t^2 + 9.8t + 73.5 = 0$? Explain why each method works.

8. Water from a hose is sprayed on a fire burning at a height of 10 m up the side of a wall. If the function $h(x) = -0.15x^2 + 3x$, where x is the horizontal distance from the fire, in metres, models the height of the water, $h(x)$, also in metres, how far back does the firefighter have to stand in order to put out the fire?

9. The president of a company that manufactures toy cars thinks that the function $P(c) = -2c^2 + 14c - 20$ represents the company's profit, where c is the number of cars produced, in thousands, and $P(c)$ is the company's profit, in hundreds of thousands of dollars. Determine the maximum profit the company can earn.

10. A ball is thrown vertically upward from the top of a cliff. The height **A** of the ball is modelled by the function $h(t) = 65 + 10t - 5t^2$, where $h(t)$ is the height in metres and t is time in seconds. Determine when the ball reaches its maximum height.

11. A computer software company models the profit on its latest video **C** game with the function $P(x) = -2x^2 + 32x - 110$, where x is the number of games the company produces, in thousands, and $P(x)$ is the profit, in thousands of dollars. How many games must the company sell to make a profit of \$16 000?
    a) Write a solution to the problem. Indicate why you chose the strategy you did.
    b) Discuss your solution and reasoning with a partner. Be ready to share your ideas with the class.

Extending

12. The population of a city, $P(t)$, is modelled by the quadratic function $P(t) = 50t^2 + 1000t + 20\,000$, where t is time in years. *Note*: $t = 0$ corresponds to the year 2000. Peg says that the population was 35 000 in 1970. Explain her reasoning for choosing that year.

13. Which pair of numbers that add to 10 will multiply to give the greatest product?

14. Jasmine and Raj have 24 m of fencing to enclose a rectangular garden. What are the dimensions of the largest rectangular garden they can enclose with that length of fencing?

3.6 Creating a Quadratic Model from Data

YOU WILL NEED
- graphing calculator
- graph paper

GOAL

Determine the equation of a curve of good fit using the factored form.

INVESTIGATE the Math

A ball is thrown into the air from the top of a building. The table of values gives the height of the ball at different times during the flight.

Time (s)	0	1	2	3	4	5
Height (m)	30	50	60	60	50	30

? What is a function that will model the data?

A. Create a scatter plot, with an appropriate scale, from the data.

B. What shape best describes the graph? Draw a curve of good fit.

curve of good fit
a curve that approximates or is close to the distribution of points in a scatter plot

C. Extend the graph to estimate the location of the zeros.

D. Use the zeros to write an equation in factored form.

E. In what direction does the parabola open? What does this tell you?

F. Using one of the points in the table, calculate the coefficient of x^2. Write the equation for the data in factored form and in standard form.

G. Using a graphing calculator and quadratic regression, determine the quadratic function model.

quadratic regression
a process that fits the second-degree polynomial $ax^2 + bx + c$ to the data

H. How does your model compare with the graphing calculator's model?

Tech Support
For help determining the equation of a curve by quadratic regression, see Technical Appendix, B-10.

Reflecting

I. How does the factored form of an equation help you determine the curve of good fit?

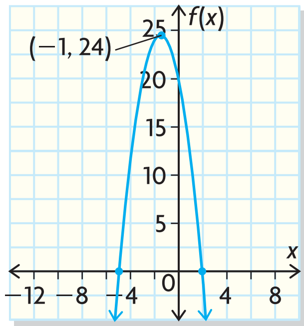
J. How will you know whether the equation is a good representation of your data?

K. How would your model change if it has only one zero? What if the model has no zeros?

APPLY the Math

EXAMPLE 1 Representing a quadratic function from its graph

Determine an equation in factored form that best represents the graph shown. Express your equation in standard form as well.



Angela's Solution: Using the Zeros and the Factored Form of a Quadratic Function

$y = a(x + 5)(x - 2)$

I used the graph to determine that the zeros of the function are −5 and 2. So I wrote the factored form of the equation.

$$
\begin{aligned}
24 &= a(-1 + 5)(-1 - 2)\\
24 &= a(4)(-3)\\
24 &= -12a\\
-2 &= a
\end{aligned}
$$

The graph passes through the point (−1, 24), so I substituted $x = -1$ and $y = 24$. Then I solved for a.

The equation in factored form is

$$
\begin{aligned}
y &= -2(x + 5)(x - 2)\\
&= -2(x^2 - 2x + 5x - 10)\\
&= -2(x^2 + 3x - 10)\\
&= -2x^2 - 6x + 20
\end{aligned}
$$

I substituted the value for a into the original equation. I expanded to get the standard form.

The equation in standard form is $y = -2x^2 - 6x + 20$.

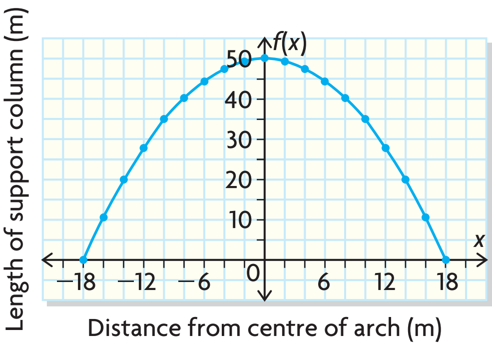
If the curve of good fit is a parabola and it crosses the x-axis, then you can estimate the zeros and use the factored form of a quadratic function to determine its equation.

EXAMPLE 2 Representing a quadratic function from data

Distance from Centre of Arch (m)	Length of Support Column (m)
−10	35.0
−8	40.0
−6	44.6
−4	47.6
−2	49.4
0	50.0
2	49.4
4	47.6
6	44.4
8	40.4
10	35.0

The track for the main hill of a roller coaster forms a parabolic arch. Vertical support columns are set in the ground to reinforce the arch every 2 m along its length. The table of values shows the length of the columns in terms of their placement relative to the centre of the arch. Negative values are to the left of the centre point. Write an algebraic model in factored form that relates the length of each column to its horizontal placement. Check your answer with a graphing calculator.

Giovanni's Solution



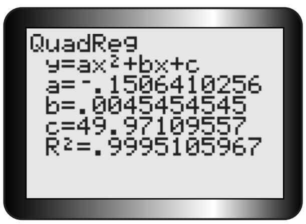
I graphed the data from the table by hand. It looks quadratic. I drew a curve of good fit and then extended the graph to identify the zeros. They're $x = 18$ and $x = -18$. I substituted those values into the factored form of the equation.

$$y = a(x - 18)(x + 18)$$
$$35 = a(10 - 18)(10 + 18)$$
$$35 = a(-8)(28)$$
$$35 = -224a$$
$$-0.15625 = a$$

The graph must pass through one of the pairs of data given in the table. I chose a point to get a. I used (10, 35). I put 10 in for x and 35 in for y in the equation, then solved for a.

$$y = -0.15625(x - 18)(x + 18)$$
$$y = -0.15625(x^2 - 324)$$
$$y = -0.15625x^2 + 50.625$$

I substituted a into my original equation and expanded into the standard form, so that I could compare my solution with the one the graphing calculator gives.



I compared my equation to the answer the calculator gave. I ignored the value of b since it was so small.

From the calculator, rounding to two decimal places, $y = -0.15x^2 + 49.97$, and from my scatter plot, $y = -0.15625x^2 + 50.625$.

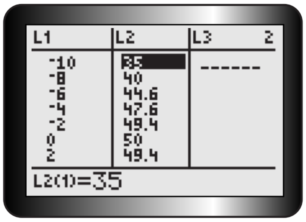
I think I did a pretty good job of finding an equation, but the graphing calculator found it much faster than I could by hand.

Tech *Support*

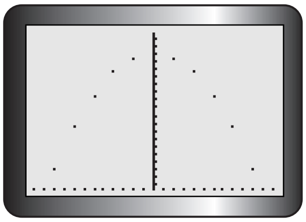
For help creating a scatter plot and determining a curve of good fit using regression, see Technical Appendix, B-10.

You can use a graphing calculator to create a scatter plot, and then use guess and check to graph the equation that fits the data.

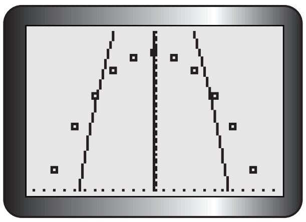
Victoria's Solution: Using the Zeros and Adjusting *a* with a Graphing Calculator



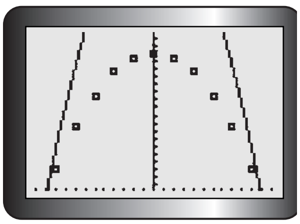
I entered the data in the lists and created a scatter plot.



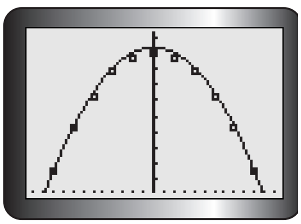
It looks quadratic. I used the quadratic equation $y = a(x - r)(x - s)$. The a-value has to be negative because the parabola opens down. The zeros of the function must be plus or minus a number greater than 10 because that is the last entry in the table of values. I tried trial and error. I tried -1 for a and ± 11 for the zeros. So the equation is $y = -1(x - 11)(x + 11)$.



That's not right because the graph isn't wide enough and doesn't go through any of the points on the scatter plot.



I tried -0.3 for a and ± 15 for the zeros, but the graph was stretched too high.



I used -0.15 because the graph must be wider and not stretched so high. I tried ± 18 for r and s because the zeros must be farther out.

I entered $y = -0.15(x - 18)(x + 18)$, which gave me a pretty good fit.

With some graphing programs you can make sliders. Sliders make it easier to change the parameters in the equation so that you can adjust and fit the curve to your data.

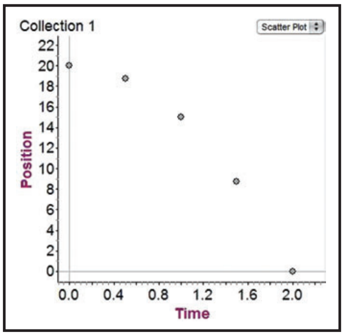
EXAMPLE 3 Representing a quadratic function with graphing software

A stone is dropped from a bridge 20 m above the water. The table of values shows the time, in seconds, and the height of the stone above the water, in metres. Write an algebraic model for the height of the stone, and use it to estimate when the stone will be 10 m above the water.

Collection 1

	1	2	3	4	5
Time (s)	0.0	0.5	1.0	1.5	2.0
Position (m)	20.00	18.75	15.00	8.75	0.00

Kommy's Solution

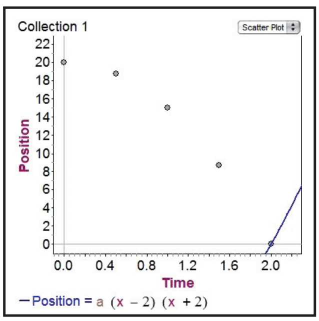


I used graphing software to create a scatter plot and see the shape.

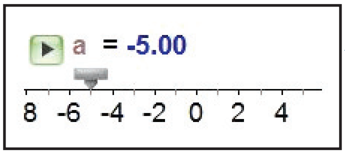
It looks quadratic because it is not a straight line but does curve downward. The maximum height happens when $x = 0$, the time the stone is dropped. One zero happens when $x = 2$, and the maximum happens halfway between the zeros. This means that the other zero is $x = -2$.

$$f(x) = a(x - 2)(x + 2)$$

I substituted the zeros into the factored form of the equation.

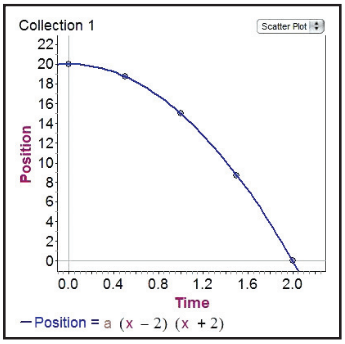


I added a slider and function to my graph. I used a slider to find the value for *a*. I tried to find a good fit by changing the value of *a*.



When $a = -5$ for the slider, the fit is pretty good.

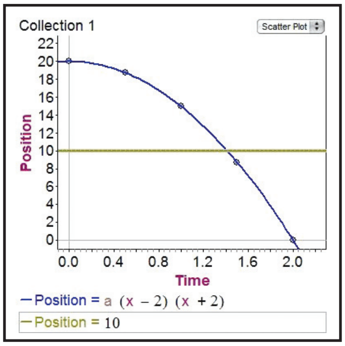
So the equation is $f(x) = -5(x - 2)(x + 2)$. I have to state the domain because the model will make sense only when $x \geq 0$, since time can't be negative.



I used my equation to find when the stone will have a height of 10 m. I set $f(x)$ equal to 10.

I expanded the left side of the equation. Since the equation has only an x^2-term and no x-term, I got the x^2-term by itself. Then I solved for x.

$$
\begin{aligned}
-5(x - 2)(x + 2) &= 10 \\
-5(x^2 - 4) &= 10 \\
-5x^2 + 20 &= 10 \\
-5x^2 &= -10 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}, x \geq 0 \\
x &\doteq 1.4
\end{aligned}
$$



I checked my answer by graphing the line $y = 10$.

Tech Support

For help with graphing software, see Technical Appendix, B-23.

In Summary

Key Ideas

- If a scatter plot of data has a parabolic shape and its curve of good fit passes through the x-axis, then the factored form of the quadratic function can be used to determine an algebraic model for the relationship.
- Once the algebraic model has been determined, it can be used to solve problems involving the relationship.

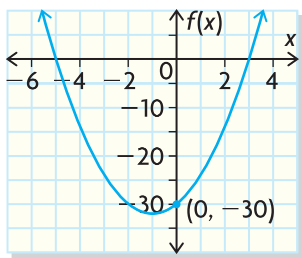
Need to Know

- The x-intercepts, or zeros, of the curve of good fit represent the values of r and s in the factored form of a quadratic function $f(x) = a(x - r)(x - s)$.
- The value of a can be determined
  - algebraically: substitute the coordinates of a point that lies on or close to the curve of good fit into $f(x)$ and solve for a.
  - graphically: estimate the value of a and graph the resulting parabola with graphing technology. By observing the graph, you can adjust your estimate of a and regraph until the parabola passes through or close to a large number of points of the scatter plot.
- Graphing technology can be used to determine the algebraic model of the curve of good fit. You can use quadratic regression if the data have a parabolic pattern.

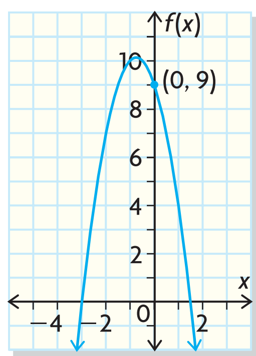
CHECK Your Understanding

1. Determine the equation of each parabola by
- determining the zeros of the function
- writing the function in factored form
- determining the value of a using a point on the curve
- expressing your answer in standard form

a)



b)



2. Create a scatter plot from the data in the table, and decide whether the data appear to be quadratic. Justify your decision.

a)

Time (h)	0	1	2	3	4	5
Population	1200	2400	4800	9600	19 200	38 400

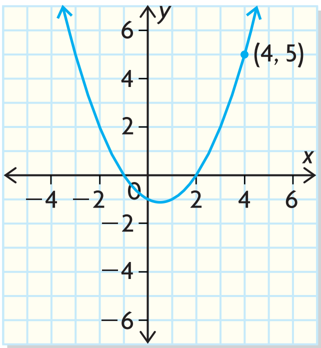
b)

Distance (m)	0	1	2	3	4	5	6	7	8
Height (m)	0	0.9	1.6	2.1	2.4	2.5	2.4	2.0	1.4

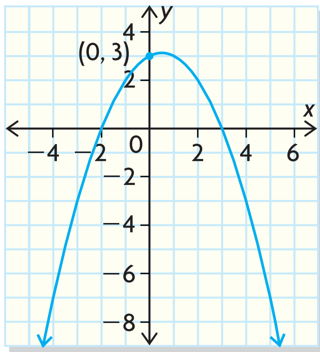
PRACTISING

3. Determine the equation of each parabola.

a)



b)



4. Write the standard form of the quadratic equation for each case.

K

	x-intercepts	*y*-intercept
a)	-3 and 4	24
b)	-2 and -5	10
c)	5 and -7	-105
d)	4 and 2	-24

5. A garden hose sprays a stream of water across a lawn. The table shows the approximate height of the stream at various distances from the nozzle. Determine an equation of the curve of good fit. Use your algebraic model to see how far away you need to stand to water a potted plant that is 1 m above the ground.

Distance from Nozzle (m)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height above Lawn (m)	0.5	1.2	1.6	1.9	2.0	1.9	1.6	1.2	0.5

Time (s)	Height (m)
0	1.0
0.5	4.5
1.0	6.0
1.5	4.5
2.0	1.0

6. While on vacation, Steve filmed a cliff diver. He analyzed the video and recorded the time and height of the diver in the table below. Determine an algebraic model to fit the data. Then use the model to predict when the diver will be 5 m above the water.

Time (s)	0	0.5	1.0	1.5	2.0
Height (m)	35.00	33.75	30.00	23.75	15.00

7. The height, at a given time, of a child above the ground when the child is on a trampoline is shown in the table. Determine an algebraic model for the data. Then use the model to predict when the child will reach a height of 3 m.

8. The height of an arrow shot by an archer is given in the table. Determine the equation of a curve of good fit. Use it to predict when the arrow will hit the ground.

t (s)	0	0.5	1.0	1.5	2.0	2.5
h (m)	0.5	8.2	13.4	16.2	16.5	14.3

9. **A** The data in the table describe the flight of a plastic glider launched from a tower on a hilltop. The height values are negative whenever the glider is below the height of the hilltop.

Time (s)	Height (m)
0	7.2
1	4.4
2	2.0
3	0
4	−1.6
5	−2.8
6	−3.6
7	−4.0
8	−4.0
9	−3.6
10	−2.8

Time (s)	Height (m)
11	−1.6
12	0
13	2.0
14	4.4
15	7.2
16	10.4
17	14.0
18	18.0
19	22.4
20	27.2

a) Write an equation to model the flight of the glider.
b) What is the lowest point in the glider's flight?

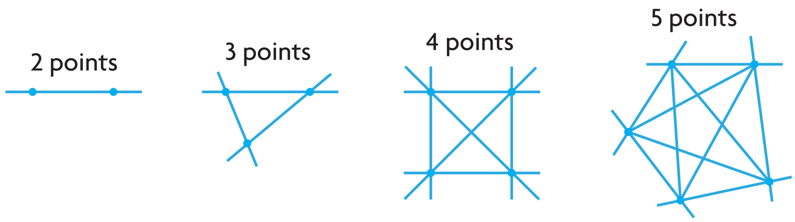
Month	Pairs of Shoes Sold
1	56
2	60
3	62
4	62
5	60
6	56
7	50
8	42
9	32
10	20
11	6

10. The Paymore Shoe Company introduced a new line of neon-green high-heeled running shoes. The table shows the number of pairs of shoes sold at one store over an 11-month period.

a) What is the equation of a curve of good fit?

b) According to your equation, when will the store sell no neon-green high-heeled running shoes?

11. T The diagrams show points joined by all possible segments.



a) Extend the pattern to include a figure with six points.

b) Write an algebraic equation for the number of line segments in terms of the number of points. Assume that the number of line segments for 0 points and 1 point is zero, since you cannot draw a line in these situations.

Number of Points	Number of Lines
0	0
1	0
2	
3	
4	
5	
6	

12. C Explain how to determine the equation of a quadratic function if the zeros of the function can be estimated from a scatter plot.

Extending

13. The Golden Gate Bridge, in San Francisco, is a suspension bridge supported by a pair of cables that appear to form parabolas. The cables are attached at either end to a pair of towers at points 152 m above the roadway. The towers are 1280 m apart, and the cable reaches its lowest point when it is 4 m above the roadway. Determine an algebraic expression that models the cable as it hangs between the towers. (*Hint:* Transfer the data to a graph such that the parabola lies below the x-axis.)

14. For a school experiment, Marcus had to record the height of a model rocket during its flight. However, during the experiment, he discovered that the motion detector he was using had stopped working. Before the detector quit, it collected the data in the table.

a) The trajectory is quadratic. Complete the table up to the time when the rocket hit the ground.

b) Determine an equation that models the height of the rocket.

c) What is the maximum height of the rocket?

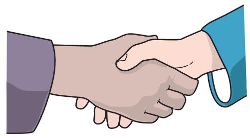
Time (s)	Height (m)
0	1.500
0.5	12.525
1.0	21.100
1.5	27.225
2.0	30.900

Curious | Math

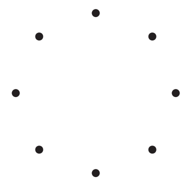
Curious Counting

YOU WILL NEED
- graphing calculator (optional)

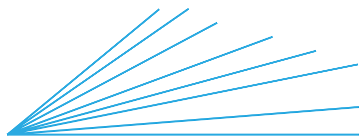
How many handshakes will there be if 8 people must shake hands with each other only once?



How many lines can be drawn that connect each pair of points if there are 8 points in total?



How many angles are there in this diagram?



How are these three problems related to each other?

1. For each problem, examine the simplest case. Determine
   - the number of handshakes if 2 people shake hands with each other only once
   - the number of lines that can be drawn that connect 2 points
   - the number of angles formed from 2 lines
2. For each problem, examine the next case by increasing the number of people, points, and lines to 3. Increase the number of people, points, and lines systematically by 1 each time, and examine each case. Record your findings in the table. What do you notice?

***n*, Number of People/ Points/Lines**	2	3	4	5	6	7	8
Number of Handshakes							
Number of Lines							
Number of Angles							

3. Create a single scatter plot using the number of people/points/lines as the independent variable and the number of handshakes/lines/angles as the dependent variable. Determine the first and second differences of the dependent variable.
4. What type of function can be used to model each situation? Explain how you know.
5. Determine the function that represents each relationship.
6. Predict how many handshakes/lines/angles there will be if 15 people/points/lines are used.
7. How are these problems related to each other?

3 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you solve a quadratic equation by factoring?

A: First, express the equation in standard form, $ax^2 + bx + c = 0$. Then use an appropriate factoring strategy to factor the quadratic expression. To determine the roots, or solutions, set each factor equal to zero and solve the resulting equations.

EXAMPLE

$$\begin{aligned} x^2 - 24 &= x + 6 \\ x^2 - x - 24 - 6 &= 0 \\ x^2 - x - 30 &= 0 \\ (x - 6)(x + 5) &= 0 \\ x - 6 = 0 \quad &\text{and} \quad x + 5 = 0 \\ x = 6 \quad &\text{and} \quad x = -5 \end{aligned}$$

Study Aid

- See Lesson 3.4, Examples 1, 2, and 4.
- Try Chapter Review Questions 6 and 7.

Q: What strategies can be used to solve problems involving quadratic functions?

A: A table of values might work, but it is time-consuming. Graphing by hand is also time-consuming. Using a graphing calculator is faster and is very helpful if the answer is not an integer. Factoring works only if the equation is written so that one side is zero and the equation can be factored. It may be difficult to tell whether the equation can be factored.

Study Aid

- See Lesson 3.5, Examples 1 and 2.
- Try Chapter Review Questions 8 and 9.

Q: What strategies can be used to create a quadratic function from data?

A: Graph the data either by hand or using a graphing calculator to see if the curve looks quadratic. If the function has zeros that are easily identifiable, use them to write the function in the form $f(x) = a(x - r)(x - s)$. Then substitute one of the points from the graph for x and $f(x)$, and solve for a. Rewrite the function, including the zeros and a. You can leave it in factored form, or you can expand the factored form to put the equation in standard form. Verify that your equation matches the data.

Study Aid

- See Lesson 3.6, Examples 1 and 2.
- Try Chapter Review Questions 10 and 11.

PRACTICE Questions

Lesson 3.2

1. Match each factored form with the correct standard form.

a) $f(x) = (x + 3)(2x - 7)$
b) $f(x) = 2(x + 7)(x - 3)$
c) $f(x) = (2x + 1)(x + 7)$
d) $f(x) = (x + 7)(x - 3)$

i) $f(x) = x^2 + 4x - 21$
ii) $f(x) = 2x^2 - x - 21$
iii) $f(x) = 2x^2 + 8x - 42$
iv) $f(x) = 2x^2 + 15x + 7$

2. Determine the maximum or minimum of each function.

a) $f(x) = x^2 - 2x - 35$
b) $f(x) = 2x^2 + 7x + 3$
c) $g(x) = -2x^2 + x + 15$

Lesson 3.3

3. Determine the zeros and the maximum or minimum value for each function.

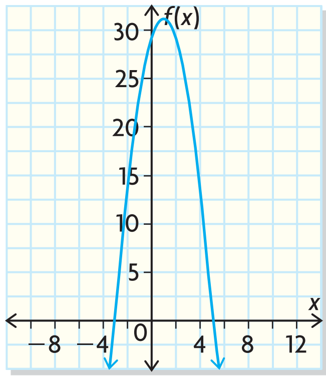
a) $f(x) = x^2 + 2x - 15$
b) $f(x) = -x^2 + 8x - 7$
c) $f(x) = 2x^2 + 18x + 16$
d) $f(x) = 2x^2 + 7x + 3$
e) $f(x) = 6x^2 + 7x - 3$
f) $f(x) = -x^2 + 49$

4. The function $h(t) = 1 + 4t - 1.86t^2$ models the height of a rock thrown upward on the planet Mars, where $h(t)$ is height in metres and t is time in seconds. Use a graph to determine

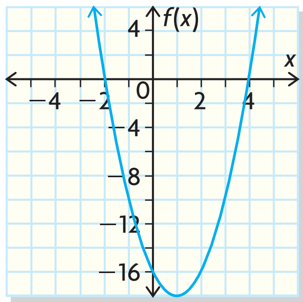
a) the maximum height the rock reaches
b) how long the rock will be above the surface of Mars

5. Determine the zeros, the coordinates of the vertex, and the y-intercept for each function.

a)



b)



Lesson 3.4

6. Solve by factoring.

a) $x^2 + 2x - 35 = 0$
b) $-x^2 - 5x = -24$
c) $9x^2 = 6x - 1$
d) $6x^2 = 7x + 5$

7. A firecracker is fired from the ground. The height of the firecracker at a given time is modelled by the function $h(t) = -5t^2 + 50t$, where $h(t)$ is the height in metres and t is time in seconds. When will the firecracker reach a height of 45 m?

Lesson 3.5

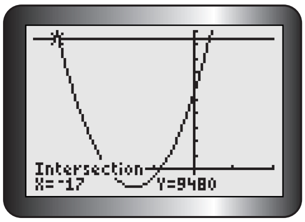
8. The population of a city, $P(t)$, is given by the function $P(t) = 14t^2 + 820t + 42\,000$, where t is time in years. *Note:* $t = 0$ corresponds to the year 2000.

a) When will the population reach 56 224? Provide your reasoning.

b) What will the population be in 2035? Provide your reasoning.

9. Fred wants to install a wooden deck around his rectangular swimming pool. The function $C(x) = 120x^2 + 1800x + 5400$ represents the cost of installation, where x is the width of the deck in metres and $C(x)$ is the cost in dollars. What will the width be if Fred spends \$9480 for the deck? Here is Steve's solution.

Steve's Solution



I used a graphing calculator to solve this problem. I entered $120x^2 + 1800x + 5400$ into Y1 and 9480 into Y2 to see where they intersect.

They intersect at two places: $x = 2$ and $x = -17$. Since both answers must be positive, use $x = 2$ and $x = 17$. Because you will get more deck with a higher number, use only $x = 17$.

Do you agree with his reasoning? Why or why not?

Lesson 3.6

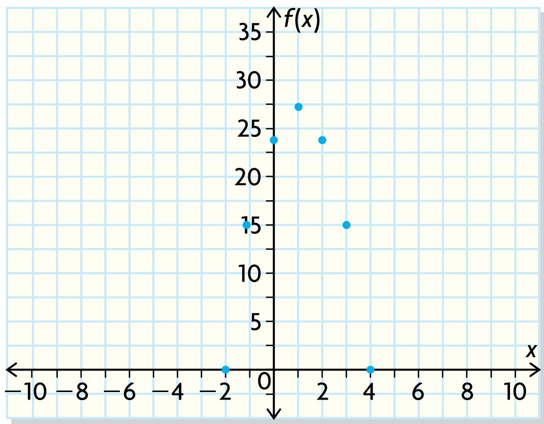
10. A toy rocket sitting on a tower is launched vertically upward. Its height y at time t is given in the table.

t Time (s)	y Height (m)
0	16
1	49
2	60
3	85
4	88
5	81
6	64
7	37
8	0

a) What is an equation of a curve of good fit?

b) How do you know that the equation in part (a) is a good fit?

11. Determine the equation of a curve of good fit for the scatter plot shown.



3 Chapter Self-Test

1. Write in standard form.
   a) $f(x) = (3x - 5)(2x - 9)$ b) $f(x) = (6 - 5x)(x + 2)$
2. Write in factored form.
   a) $f(x) = x^2 - 81$ b) $f(x) = 6x^2 + 5x - 4$
3. Determine the zeros, the axis of symmetry, and the maximum or minimum value for each function.
   a) $f(x) = x^2 + 2x - 35$ b) $f(x) = -4x^2 - 12x + 7$
4. Can all quadratic equations be solved by factoring? Explain.
5. Solve by graphing.
   a) $x^2 + 6x - 3 = -3$ b) $2x^2 - 5x = 7$
6. Solve by factoring.
   a) $2x^2 + 11x - 6 = 0$ b) $x^2 = 4x + 21$
7. The population of a town, $P(t)$, is modelled by the function $P(t) = 6t^2 + 110t + 3000$, where t is time in years. *Note:* $t = 0$ represents the year 2000.
   a) When will the population reach 6000?
   b) What will the population be in 2030?
8. Target-shooting disks are launched into the air from a machine 12 m above the ground. The height, $h(t)$, in metres, of the disk after launch is modelled by the function $h(t) = -5t^2 + 30t + 12$, where t is time in seconds.
   a) When will the disk reach the ground?
   b) What is the maximum height the disk reaches?
9. Students at an agricultural school collected data showing the effect of different annual amounts of rainfall, x, in hectare-metres (ha · m), on the yield of broccoli, y, in hundreds of kilograms per hectare (100 kg/ha). The table lists the data.
   a) What is an equation of the curve of good fit?
   b) How do you know whether the equation in part (a) is a good fit?
   c) Use your equation to calculate the yield when there is 1.85 ha · m of annual rainfall.

Rainfall (ha · m)	Yield (100 kg/ha)
0.30	35
0.45	104
0.60	198
0.75	287
0.90	348
1.05	401
1.20	427
1.35	442
1.50	418

10. Why is it useful to have a curve of good fit?

Quadratic Cases

Quadratic functions are used as mathematical models for many real-life situations, such as designing architectural supports, decorative fountains, and satellite dishes. Quadratic models can show trends and provide predictions about relationships among data. They can also be used to solve problems involving maximizing profit, minimizing the amount of material used to manufacture something, and calculating the location of a projectile, such as a ball.

? What can you model with a quadratic function?

A. Collect some data for which the zeros can be determined easily, from either a secondary source, such as the Internet, or an experiment that can be modelled with a quadratic function.

B. Determine a model in factored form that can be used to represent your data.

C. Create several questions that can be answered by your model.

D. In a report, indicate
- how you decided that a quadratic model would be appropriate
- how you manually determined the factored form of the equation that fits your data
- how well you think the equation fits your data
- the questions you created and the answers you determined by using the model

Task Checklist

✔ Did you show all your steps?

✔ Did you include a graph?

✔ Did you support your choice of data?

✔ Did you explain your thinking clearly?