$y^2 = 4x$

$2t\frac{dy}{dx} = 4$

At $(1, 2)$, $\frac{dy}{dx} = 1$.

Therefore, the slope of the tangent line at $(1, 2)$ is 1 and the equation of the normal is

$\frac{y - 2}{x - 1} = -1$ or $x + y = 3$.

The centres of the two circles lie on the straight line $x + y = 3$. Let the coordinates of the centre of each circle be $(p, q) = (p, 3 - p)$. The radius of each circle is $3\sqrt{2}$. Since $(1, 2)$ is on the circumference of the circles,

$$(p - 1)^2 + (3 - p - 2)^2 = r^2$$
$$p^2 - 2p + 1 + 1 - 2p + p^2 = (3\sqrt{2})^2$$
$$2p^2 - 4p + 2 = 18$$
$$p^2 - 2p - 8 = 0$$
$$(p - 4)(p + 2) = 0$$

$p = 4$ or $p = -2$

$q = -1$ or $q = 5$.

Therefore, the centres of the circles are $(-2, 5)$ and $(4, -1)$. The equations of the circles are

$(x + 2)^2 + (y - 5)^2 = 18$ and

$(x - 4)^2 + (y + 1)^2 = 18$.

Related Rates, pp. 569–570

1. a. $\frac{dA}{dt} = 4 \text{ m/s}^2$

b. $\frac{dS}{dt} = -3 \text{ m}^2\text{/min.}$

c. $\frac{ds}{dt} = 70$ km/h when $t = 0.25$

d. $\frac{dx}{dt} = \frac{dy}{dt}$

e. $\frac{d\theta}{dt} = \frac{\pi}{10}$ rad/s

2. $T(x) = \frac{200}{1 + x^2}$

a. $\frac{dx}{dt} = 2$ m/s

Find $\frac{dT(x)}{dt}$ when $x = 5$ m:

$$T(x) = \frac{200}{1 + x^2}$$
$$= 200(1 + x^2)^{-1}$$

$$\frac{dT(x)}{dt} = -200(1 + x^2)^{-2}2x\frac{dx}{dt}$$
$$= \frac{-400x}{(1 + x^2)^2} \cdot \frac{dx}{dt}.$$

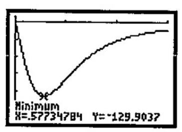
At a specific time, when $x = 5$,

$$\frac{dT(5)}{dt} = \frac{-400(5)}{(26)^2} \quad (2)$$
$$= \frac{-4000}{676}$$
$$= \frac{-1000}{169}$$
$$\frac{dT(5)}{dt} \doteq -5.9.$$

Therefore, the temperature is decreasing at a rate of 5.9 °C/s.

b. To determine the distance at which the temperature is changing fastest, graph the derivative of the function on a graphing calculator. Since the temperature decreases as the person moves away from the fire, the temperature will be changing fastest when the value of the derivative is at its minimum value.

For Y1, enter nDeriv(from the MATH menu. Then enter 200 [key] (1+X²), X, X).



The temperature is changing fastest at about 0.58 m.

c. Solve $T''(x) = 0$.

$$T'(x) = \frac{-400x}{(1 + x^2)^2}$$

$$T'(x) = \frac{-400(1 + x^2)^2 - 2(1 + x^2)(2x)(-400x)}{(1 + x^2)^4}$$

Let $T''(x) = 0$,

$-400(1 + x^2)^2 + 1600x^2(1 + x^2) = 0$.

Divide,

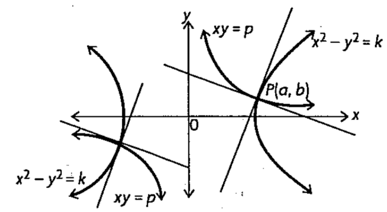
$$400(1 + x^2) - (1 + x^2) + 4x^2 = 0$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{\sqrt{3}}$$
$$x = \frac{1}{\sqrt{3}}$$

$x > 0$ or $x \doteq 0.58$.

$$\frac{dy}{dx} = \frac{\frac{16}{5}}{4\left(\frac{6}{5}\right)} \text{ or } f(q) = 0$$
$$= \frac{2}{3}$$

Equation of the tangent at Q is $\frac{y - 6}{x - 4} = \frac{2}{3}$ or $2x - 3y + 10 = 0$ or equation of tangent at A is $x = 4$.

14.



Let $P(a, b)$ be the point of intersection where $a \neq 0$ and $b \neq 0$.

For $x^2 - y^2 = k$

$$2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{x}{y}$$

At $P(a, b)$,

$\frac{dy}{dx} = \frac{a}{b}$.

For $xy = P$,

$$1 \cdot y + \frac{dy}{dx}x = P$$
$$\frac{dy}{dx} = -\frac{y}{x}$$

At $P(a, b)$,

$\frac{dy}{dx} = -\frac{b}{a}$.

At point $P(a, b)$, the slope of the tangent line of $xy = P$ is the negative reciprocal of the slope of the tangent line of $x^2 - y^2 = k$. Therefore, the tangent lines intersect at right angles, and thus, the two curves intersect orthogonally for all values of the constants k and P.

15. $\sqrt{x} + \sqrt{y} = \sqrt{k}$

Differentiate with respect to x:

$$\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}y^{\frac{1}{2}}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Let $P(a, b)$ be the point of tangency.

$\frac{dy}{dx} - \frac{\sqrt{b}}{\sqrt{a}}$

Equation of tangent line l at P is

$\frac{y - b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}$.

x-intercept is found when $y = 0$.

$$\frac{-b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}$$
$$-b\sqrt{a} = -\sqrt{b}x + a\sqrt{b}$$
$$x = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$$

Therefore, the x-intercept is $\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$.

For the y-intercept, let $x = 0$,

$\frac{y - b}{-a} = -\frac{\sqrt{b}}{\sqrt{a}}$.

y-intercept is $\frac{a\sqrt{b}}{\sqrt{a}} + b$.

The sum of the intercepts is

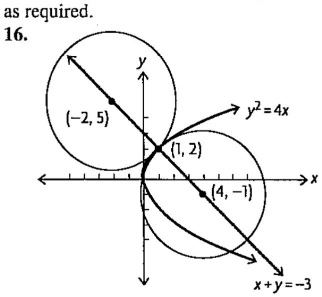
$$\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}} + \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a}}$$
$$= \frac{a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab + b^{\frac{3}{2}}a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$
$$= \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}(a + 2\sqrt{a}\sqrt{b} + b)}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$
$$= a + 2\sqrt{a}\sqrt{b} + b$$
$$= (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$$

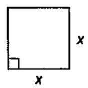
Since $P(a, b)$ is on the curve, then

$\sqrt{a} + \sqrt{b} = \sqrt{k}$, or $a^{\frac{1}{2}} + b^{\frac{1}{2}} = k^{\frac{1}{2}}$.

Therefore, the sum of the intercepts is $= k$, as required.

16.



3. Given square  $\frac{dx}{dt} = 5$ cm/s.

Find $\frac{dA}{dt}$ when $x = 10$ cm.

Solution

Let the side of a square be x cm.

$A = x^2$

$\frac{dA}{dt} = 2x\frac{dx}{dt}$

At a specific time, $x = 10$ cm.

$\frac{dA}{dt} = 2(10)(5)$

$= 100$

Therefore, the area is increasing at 100 cm^2/s when a side is 10 cm.

$P = 4x$

$\frac{dP}{dt} = 4\frac{dx}{dt}$

At any time, $\frac{dx}{dt} = 5$.

$\frac{dP}{dt} = 20.$

Therefore, the perimeter is increasing at 20 cm/s.

4. Given cube with sides x cm, $\frac{dx}{dt} = 5$ cm/s.

a. Find $\frac{dv}{dt}$ when $x = 5$ cm:

$V = x^3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

At a specific time, $x = 5$ cm.

$\frac{dV}{dt} = 3(5)^2(4)$

$= 300$

Therefore, the volume is increasing at 300 cm^3/s.

b. Find $\frac{dS}{dt}$ when $x = 7$ cm.

$S = 6x^2$

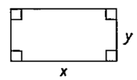
$\frac{dS}{dt} = 12x\frac{dx}{dt}$

At a specific time, $x = 7$ cm,

$\frac{dS}{dt} = 12(7)(4)$

$= 336.$

Therefore, the surface area is increasing at a rate of 336 cm^2/s.

5. Given rectangle 

$\frac{dx}{dt} = 2$ cm/s

$\frac{dy}{dt} = -3$ cm/s

Find $\frac{dA}{dt}$ when $x = 20$ cm and $y = 50$ cm.

Solution

$A = xy$

$\frac{dA}{dt} = \frac{dx}{dt}y + \frac{dy}{dt}x$

At s specific time, $x = 20$, $y = 50$,

$\frac{dA}{dt} = (2)(50) + (-3)(20)$

$= 100 - 50$

$= 40.$

Therefore, the area is increasing at a rate of 40 cm^2/s.

6. Given circle with radius r,

$\frac{dA}{dt} = -5\ m^2/s.$

a. Find $\frac{dr}{dt}$ when $r = 3$ m.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 3$,

$-5 = 2\pi(3)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-5}{6\pi}$

Therefore, the radius is decreasing at a rate of $\frac{5}{6\pi}$ m/s when $r = 3$ m.

b. Find $\frac{dD}{dt}$ when $r = 3$.

$\frac{dD}{dt} = 2\frac{dr}{dt}$

$= 2\left(\frac{-5}{6\pi}\right)$

$= \frac{-5}{3\pi}$

Therefore, the diameter is decreasing at a rate of $\frac{5}{3\pi}$ m/s.

7. Given circle with radius r, $\frac{dA}{dt} = 6\ km^2/h$

Find $\frac{dr}{dt}$ when $A = 9\pi\ km^2$.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $A = 9\pi$,

$9\pi = \pi r^2$

$r^2 = 9$

$r = 3$

$r > 0.$

When $r = 3$,

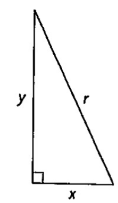
$\frac{dA}{dt} = 6$

$6 = 2\pi(3)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{\pi}.$

Therefore, the radius is increasing at a rate of $\frac{1}{\pi}$ km/h.

8. Let x represent the distance from the wall and y the height of the ladder on the wall.



$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

When $r = 5$, $y = 3$,

$x^2 = 25 - 9$

$= 16$

$x = 4$

$x = 4, y = 3, r = 5$

$\frac{dx}{dt} = \frac{1}{3}, \frac{dr}{dt} = 0.$

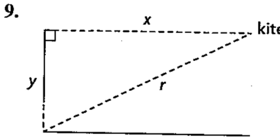
Substituting,

$4\left(\frac{1}{3}\right) + 3\left(\frac{dy}{dt}\right) = 5(0)$

$\frac{dy}{dt} = -\frac{4}{9}.$

Therefore, the top of the ladder is sliding down at 4 m/s.

9.



Let the variables represent the distances as shown on the diagram.

$x^2 + y^2 = r^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

$x = 30, y = 40$

$r^2 = 30^2 + 40^2$

$r = 50$

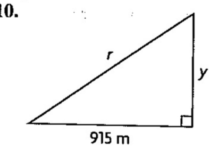
$\frac{dr}{dt} = ?, \frac{dx}{dt} = 10, \frac{dy}{dt} = 0$

$30(10) + 40(0) = 50\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = 8$

Therefore, she must let out the line at a rate of 8 m/min.

10.



Label diagram as shown.

$r^2 = y^2 + 915^2$

$2r\frac{dr}{dt} = 2y\frac{dy}{dt}$

$r\frac{dr}{dt} = y\frac{dy}{dt}$

When $y = 1220$, $\frac{dy}{dt} = 268$ m/s.

$r = \sqrt{1220^2 + 915^2}$

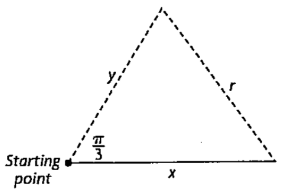
$= 1525$

$1525\left(\frac{dr}{dt}\right) = 1220 \times 268$

$\frac{dr}{dt} = 214$ m/s

Therefore, the camera–to–rocket distance is changing at 214 m/s.

11.



$\frac{dx}{dt} = 15$ km/h

$\frac{dy}{dt} = 20$ km/h

Find $\frac{dr}{dt}$ when $t = 2$ h.

Solution

Let x represent the distance cyclist 1 is from the starting point, $x \geq 0$. Let y represent the distance cyclist 2 is from the starting point, $y \geq 0$ and let r be the distance the cyclists are apart. Using the cosine law,

$$r^2 = x^2 + y^2 - 2xy\cos\frac{\pi}{3}$$
$$= x^2 + y^2 - 2xy\left(\frac{1}{2}\right)$$
$$r^2 = x^2 + y^2 - xy$$
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left[\frac{dx}{dt}y + \frac{dy}{dt}x\right]$$

At $t = 2$ h, $x = 30$ km, $y = 40$ km

and $r^2 = 30^2 + 40^2 - 2(30)(40)\cos\frac{\pi}{3}$

$$= 2500 - 2(1200)\left(\frac{1}{2}\right)$$
$$= 1300$$
$$r = 10\sqrt{13},\ r > 0.$$
$$\therefore 2(10\sqrt{13})\frac{dr}{dt}$$
$$= 2(30)(15) + 2(40)(20) - [15(40) + 20(30)]$$
$$20\sqrt{13}\frac{dr}{dt} = 900 + 1600 - [600 - 600]$$
$$= 1300$$
$$\frac{dr}{dt} = \frac{130}{2\sqrt{13}}$$
$$= \frac{65}{\sqrt{13}}$$
$$= \frac{65\sqrt{13}}{13}$$
$$= 5\sqrt{13}$$

Therefore, the distance between the cyclists is increasing at a rate of $5\sqrt{13}$ km/h after 2 h.

12. Given sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 8$ cm^3/s.

a. Find $\frac{dr}{dt}$ when $r = 12$ cm.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

At a specific time, when $r = 12$ cm:

$$8 = 4\pi(12)^2\frac{dr}{dt}$$
$$8 = 4\pi(144)\frac{dr}{dt}$$
$$\frac{1}{72\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{72\pi}$ cm/s.

b. Find $\frac{dr}{dt}$ when $V = 1435$ cm^3.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

At a specific time, when $V = 1435$ cm^3:

$$V = 1435$$
$$\frac{4}{3}\pi r^3 = 1435$$
$$r^3 \doteq 342.581\ 015$$
$$\doteq 6.997\ 148\ 6$$
$$= 7$$
$$8 \doteq 4\pi(7)^2\frac{dr}{dt}$$
$$8 = 196\pi\frac{dr}{dt}$$
$$\frac{2}{49\pi} = \frac{dr}{dt}$$
$$0.01 = \frac{dr}{dt}.$$

Therefore, the radius is increasing at $\frac{2}{49\pi}$ cm/s (or about 0.01 cm/s).

c. Find $\frac{dr}{dt}$ when $t = 33.5$ s.

When $t = 33.5$, $v = 8 \times 33.5$ cm^3:

$$\frac{4}{3}\pi r^3 = 268$$
$$r^3 \doteq 63.980\ 287\ 12$$

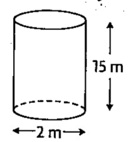
$$r \doteq 3.999\ 589\ 273$$
$$\doteq 4.$$
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

At $t = 33.5$ s,

$$8 \doteq 4\pi(4)^2\frac{dr}{dt}$$
$$8 = 64\pi\frac{dr}{dt}$$
$$\frac{1}{8\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{8\pi}$ cm/s (or 0.04 cm/s).

13. Given cylinder



$$V = \pi r^2 h$$
$$\frac{dV}{dt} = 500 \text{ L/min}$$
$$= 500\ 000 \text{ cm/min}$$

Find $\frac{dh}{dt}$.

$V = \pi r^2 h$

Since the diameter is constant at 2 m, the radius is also constant at 1 m = 100 cm.

$$V = 10\ 000\pi h$$
$$\frac{dV}{dt} = 10\ 000\pi\frac{dh}{dt}$$
$$500\ 000 = 10\ 000\pi\frac{dh}{dt}$$
$$\frac{50}{\pi} = \frac{dh}{dt}$$

Therefore, the fluid level is rising at a rate of $\frac{50}{\pi}$ cm/min.

Find t, the time of fill the cylinder.

$V = \pi r^2 h$

$V = \pi(100)^2(1\ 500)$ cm^3

$V = 150\ 000\ 00\pi$ cm^3

Since $\frac{dV}{dt} = 500\ 000$ cm^3/min, it takes $\frac{15\ 000\ 000\pi}{500\ 000}$ min,

$= 30\pi$ min to fill

$\doteq 94.25$ min.

Therefore, it will take 94.25 min, or just over 1.5 h to fill the cylindrical tank.

14. There are many possible problems.

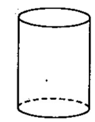
Samples:

a. The diameter of a right-circular cone is expanding at a rate of 4 cm/min. Its height remains constant at 10 cm. Find its radius when the volume is increasing at a rate of 80π cm^3/min.

b. Water is being poured into a right-circular tank at the rate of 12π m^3/min. Its height is 4 m and its radius is 1 m. At what rate is the water level rising?

c. The volume of a right-circular cone is expanding because its radius is increasing at 12 cm/min and its height is increasing at 6 cm/min. Find the rate at which its volume is changing when its radius is 20 cm and its height is 40 cm.

15. Given cylinder



$d = 1$ m

$h = 15$ m

$\frac{dr}{dt} = 0.003$ m/year

$\frac{dh}{dt} = 0.4$ m/year

Find $\frac{dV}{dt}$ at the instant $D = 1$

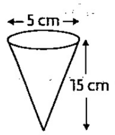
$$V = \pi r^2 h$$
$$\frac{dV}{dt} = \left(2\pi r\frac{dr}{dt}\right)(h) + \left(\frac{dh}{dt}\right)(\pi r^2).$$

At a specific time, when $D = 1$; i.e., $r = 0.5$,

$$\frac{dV}{dt} = 2\pi(0.5)(0.003)(15) + 0.4\pi(0.5)^2$$
$$= 0.045\pi + 0.1\pi$$
$$= 0.145\pi$$

Therefore, the volume of the trunk is increasing at a rate of 0.145π m^3/year.

16. Given cone



$r = 5$ cm
$h = 15$ cm
$\frac{dV}{dt} = 2 \text{ cm}^3/\text{min}$
Find $\frac{dh}{dt}$ when $h = 3$ cm,
$V = \frac{1}{3}\pi r^2 h.$
Using similar triangles, $\frac{r}{h} = \frac{5}{15} = \frac{1}{3}$
$r = \frac{h}{3}.$
Substituting into $V = \frac{1}{3}\pi r^2 h$,

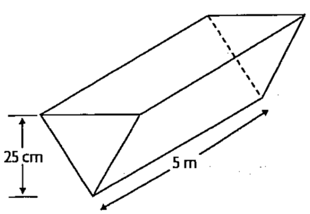
$$V = \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$$
$$= \frac{1}{27}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

At a specific time, when $h = 3$ cm,

$$-2 = \frac{1}{9}\pi(3)^2\frac{dh}{dt}$$
$$-\frac{2}{\pi} = \frac{dh}{dt}.$$

Therefore, the water level is being lowered at a rate of $\frac{2}{\pi}$ cm/min when height is 3 cm.

17. Given trough



$\frac{dV}{dt} = 0.25\frac{\text{m}^3}{\text{min}}$
Find $\frac{dh}{dt}$ when $h = 10$ cm
$= 0.1$ m.

Since the cross section is equilateral, $V = \frac{h^2}{\sqrt{3}} \times l.$

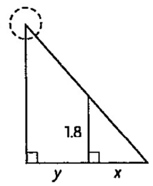
$$V = \frac{h^2}{\sqrt{3}} \times 5.$$
$$\frac{dV}{dt} = \frac{10}{\sqrt{3}}h\frac{dh}{dt}$$

At a specific, time when $h = 0.1 = \frac{1}{10}$,

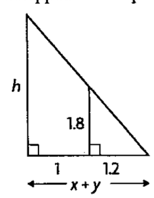
$$0.25 = \frac{10}{\sqrt{3}}\frac{1}{10}\frac{dh}{dt}$$
$$0.25\sqrt{3} = \frac{dh}{dt}$$
$$\frac{\sqrt{3}}{4} = \frac{dh}{dt}$$

Therefore, the water level is rising at a rate of $\frac{\sqrt{3}}{4}$ m/min.

18.



$\frac{dy}{dt} = 120$ m/min
Find $\frac{dx}{dt}$ when $t = 5$ s.
Let x represent the length of the shadow. Let y represent the distance the man is from the base of the lamppost. Let h represent the height of the lamppost. At a specific instant, we have



Using similar triangles,

$$\frac{x + y}{h} = \frac{1.2}{1.8}$$
$$\frac{2.2}{h} = \frac{2}{3}$$
$$2h = 6.6$$
$$h = 3.3$$

Therefore, the lamppost is 3.3 m high.

At any time,

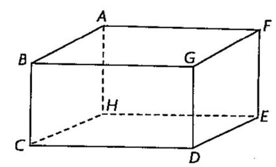
$$\frac{x + y}{x} = \frac{3.3}{1.8}$$
$$\frac{x + y}{x} = \frac{11}{6}$$
$$6x + 6y = 11x$$
$$6y = 5x$$
$$6\frac{dy}{dt} = 5\frac{dx}{dt}.$$

At a specific time, when $t = 5$ seconds
$\frac{dy}{dt} = 120$ m/min,

$$6 \times 120 = 5\frac{dx}{dt}$$
$$\frac{dx}{dt} = 144.$$

Therefore, the man's shadow is lengthening at a rate of 144 m/min after 5 s.

19. This question is similar to finding the rate of change of the length of the diagonal of a rectangular prism.



$$20\text{m} = \frac{20}{1000}\text{km}$$
$$= \frac{1}{50}\text{km}$$

Find $\frac{d(GH)}{dt}$ at $t = 10$ s,

$$= \frac{1}{360}\text{ h.}$$

Let BG be the path of the train and CH be the path of the boat:

$$\frac{d(BG)}{dt} = 60 \text{ km/h and } \frac{d(CH)}{dt} = 20 \text{ km/h.}$$

At $t = \frac{1}{360}$ h, $BG = 60 \times \frac{1}{360}$

$$= \frac{1}{6}\text{ km}$$

and $CH = 20 \times \frac{1}{360}$

$$= \frac{1}{18}\text{ km.}$$

Using the Pythagorean theorem,
$GH^2 = HD^2 + DG^2$ and $HD^2 = CD^2 + CH^2$
$GH^2 = CD^2 + CH^2 + DG^2$
Since $BG = CD$ and $FE = GD = \frac{1}{50}$, it follows that

$$GH^2 = BG^2 + CH^2 + \frac{1}{2500}.$$
$$2(GH)\frac{d(GH)}{dt} = 2(BG)\frac{d(BG)}{dt} + 2(CH)\frac{d(CH)}{dt}$$

At $t = 10$ s,

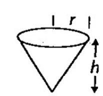
$$GH\frac{d(GH)}{dt} = \frac{1}{6}(60) + \frac{1}{18}(20)$$
$$\frac{\sqrt{6331}}{450}\frac{d(GH)}{dt} = \frac{100}{9}$$
$$\frac{d(GH)}{dt} = \frac{45000}{9\sqrt{6331}}$$
$$\doteq 62.8.$$

And $GH^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{50}\right)^2$

$$= \frac{1}{36} + \frac{1}{324} + \frac{1}{2500}$$
$$= \frac{911\,664}{29\,160\,000} + 8$$
$$GH^2 = \frac{113\,958}{364\,500} + 18$$
$$= \frac{6331}{202\,500}$$
$$GH = \frac{\sqrt{6331}}{450} = \frac{\sqrt{13 \times 487}}{450}$$

Therefore, they are separating at a rate of approximately 62.8 km/h.

20. Given cone



a. $r = h$

$$\frac{dV}{dt} = 200 - 20$$
$$= 180 \text{ cm}^3/\text{s}$$

Find $\frac{dh}{dt}$ when $h = 15$ cm.

$$V = \frac{1}{3}\pi r^2 h \text{ and } r = h$$
$$V = \frac{1}{3}\pi h^3.$$
$$\frac{dV}{dt} = \pi h^2\frac{dh}{dt}$$

At a specific time, $h = 15$ cm.

$180 = \pi(15)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{5\pi}$

Therefore, the height of the water in the funnel is increasing at a rate of $\frac{4}{5\pi}$ cm/s.

b. $\frac{dV}{dt} = 200\ \text{cm}^3/\text{s}$

Find $\frac{dh}{dt}$ when $h = 25$ cm.

$\frac{dV}{dt} = \pi h^2\frac{dh}{dt}$.

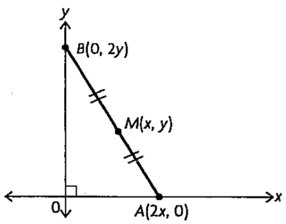
At the time when the funnel is clogged, $h = 25$ cm:

$200 = \pi(25)^2\frac{dh}{dt}$.

$\frac{dh}{dt} = \frac{8}{25\pi}$.

Therefore, the height is increasing at $\frac{8}{25\pi}$ cm/s.

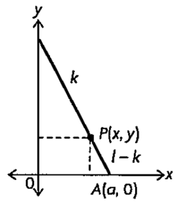
21.



Let the midpoint of the ladder be (x, y). From similar triangles, it can be shown that the top of the ladder and base of the ladder would have points $B(0, 2y)$ and $A(2x, 0)$ respectively. Since the ladder has length l,

$$(2x)^2 + (2y)^2 = l^2$$
$$4x^2 + 4y^2 = l^2$$
$$x^2 + y^2 = \frac{l^2}{4}$$
$$= \left(\frac{l}{2}\right)^2 \text{ is the required equation.}$$

Therefore, the equation of the path followed by the midpoint of the ladder represents a quarter circle with centre (0, 0) and radius $\frac{l}{2}$, with $x, y \geq 0$.



Let $P(x, y)$ be a general point on the ladder a distance k from the top of the ladder. Let $A(a, 0)$ be the point of contact of the ladder with the ground.

From similar triangles, $\frac{a}{l} = \frac{x}{k}$ or $a = \frac{xl}{k}$.

Using the Pythagorean Theorem:

$y^2 + (a - x)^2 = (l - k)^2$, and substituting $a = \frac{xl}{k}$,

$$y^2 + \left(\frac{xl}{k} - x\right)^2 = (l - k)^2$$
$$y^2 + x^2\left(\frac{l - k}{k}\right)^2 = (l - k)^2$$
$$\frac{(l - k)^2}{k^2}x^2 + y^2 = (l - k)^2$$
$$\frac{x^2}{k^2} + \frac{y^2}{(l - k)^2} = 1 \text{ is the required equation.}$$

Therefore, the equation is the first quadrant portion of an ellipse.

The Natural Logarithm and its Derivative, p. 575

1. A natural logarithm has base e; a common logarithm has base 10.

2. Since $e = \lim_{h\to 0}(1 + h)^{\frac{1}{h}}$, let $h = \frac{1}{n}$. Therefore,

$$e = \lim_{\frac{1}{n}\to 0}\left(1 + \frac{1}{n}\right)^n,$$

But as $\frac{1}{n} \to 0$, $n \to \infty$.

Therefore, $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

If $n = 100$, $e \doteq \left(1 + \frac{1}{100}\right)^{100}$

$= 1.01^{100}$

$\doteq 2.70481$.

Try $n = 100\,000$, etc.